

ON THE MOMENTS OF THE CHARACTERISTIC POLYNOMIAL OF A GINIBRE RANDOM MATRIX

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ABSTRACT. In this article we study the large N asymptotics of small fractional moments of the absolute value of the characteristic polynomial of a $N \times N$ complex Ginibre random matrix with the characteristic polynomial evaluated at a point in the unit disk. More precisely, we calculate the large N asymptotics of $\mathbb{E}|\det(G_N - z)|^\gamma$, where G_N is a $N \times N$ matrix whose entries are i.i.d and distributed as $N^{-1}Z$, Z being a standard complex Gaussian, $\gamma \in (0, 2)$, and $|z| < 1$. This expectation is proportional to the determinant of a complex moment matrix with a symbol which is supported in the whole complex plane and has a Fisher-Hartwig type of singularity: $\det(\int_{\mathbb{C}} w^i \bar{w}^j |w - z|^\gamma e^{-N|w|^2} d^2w)_{i,j=0}^{N-1}$. We study the asymptotics of this determinant using recent results due to Lee and Yang concerning the asymptotics of orthogonal polynomials with respect to the weight $|w - z|^\gamma e^{-N|w|^2} d^2w$ [21] along with differential identities familiar from the study of asymptotics of Toeplitz and Hankel determinants with Fisher-Hartwig singularities [10, 12, 18]. To our knowledge, even in the case of one singularity, the asymptotics of the determinant of such a moment matrix whose symbol has support in a two-dimensional set and a Fisher-Hartwig singularity, have been previously unknown.

1. INTRODUCTION AND MAIN RESULT

The goal of this article is to study the large N asymptotics of moments of the absolute value of the characteristic polynomial of a $N \times N$ complex Ginibre random matrix, with the characteristic polynomial evaluated at a fixed point in the unit disk. More precisely, we prove the following result:

Theorem 1.1. *Let G_N be a $N \times N$ complex Ginibre random matrix (i.e. its entries are i.i.d. and distributed as $N^{-1}Z$, where Z is a standard complex Gaussian), $\gamma \in (0, 2)$, and $z \in \mathbb{C}$ with $|z| < 1$. Then as $N \rightarrow \infty$*

$$\mathbb{E}|\det(G_N - z)|^\gamma = (1 + o(1))N^{\frac{\gamma^2}{8}} e^{\frac{\gamma}{2}N(|z|^2 - 1)} \frac{(2\pi)^{\frac{\gamma}{4}}}{G(1 + \frac{\gamma}{2})},$$

where G is the Barnes G -function, and the error is uniform in $\gamma \in [0, \gamma_0]$ for fixed $\gamma_0 \in (0, 2)$ and uniform in $z \in \{w \in \mathbb{C} : r \leq |w| \leq R\}$ with fixed $0 < r \leq R < 1$.

In the remainder of this introduction, we'll briefly discuss some motivation and interpretations of this result as well as give an outline of the rest of the article.

1.1. Motivation – moment matrices with Fisher-Hartwig singularities and random geometry. In addition to the direct application of giving information about the spectrum of the matrix G_N , understanding moments of the form $\mathbb{E} \prod_{j=1}^k |\det(G_N - z_j)|^{\gamma_j}$ is interesting due to connections to problems in various areas of mathematics. Let us first point out that if one were considering the case where G_N was replaced by a Haar distributed unitary matrix (the circular unitary ensemble), such moments can be expressed as Toeplitz determinants whose symbol has so-called Fisher-Hartwig singularities. The large N asymptotics of such determinants has a relatively long and interesting history – see e.g. [10, 11, 12] for background and recent results concerning the problem. In the case where the matrix G_N is replaced by a random Hermitian matrix such as a GUE matrix, such asymptotics have again been successfully studied through a connection to the asymptotics of Hankel determinants with Fisher-Hartwig singularities – see e.g. [18, 6].

As we will recall in Section 2, also moments of the form $\mathbb{E} \prod_{j=1}^k |\det(G_N - z_j)|^{\gamma_j}$ can be expressed in terms of determinants of moment matrices, but now of the form $\det(\int_{\mathbb{C}} w^i \bar{w}^j \prod_{l=1}^k |w - z_l|^{\gamma_l} e^{-N|w|^2} d^2 w)_{i,j=0}^{N-1}$. Despite the success in the case of Haar distributed unitary matrices and random Hermitian matrices, to our knowledge, there are virtually no results concerning the asymptotics of determinants of such “fully complex” moment matrices with Fisher-Hartwig singularities (though we refer to [16, Corollary 2], where a representation of even integer moments of the characteristic polynomial in terms of matrix hypergeometric functions is obtained). From this point of view, Theorem 1.1 can be seen as a first step in the direction of a Fisher-Hartwig formula for such two-dimensional symbols.

Further motivation for Theorem 1.1 comes from random geometry. In [26], Rider and Virág proved a central limit theorem for linear statistics of the Ginibre ensemble (i.e. for $\text{Tr}(f(G_N))$ for suitable functions f) and pointed out that this is equivalent to $\log |\det(G_N - z)| - \mathbb{E} \log |\det(G_N - z)|$ converging to a variant of the Gaussian free field in a suitable sense. The limiting object here can be understood as a random generalized function which is formally a Gaussian process whose correlation kernel is $-\frac{1}{2} \log |z - w|$ for z, w in the unit disk. Such random generalized functions have recently been discovered to be closely related to conformally invariant SLE-type random curves as well as the scaling limits of random planar maps – see e.g. [2, 5, 7, 22, 27].

In this connection between the Gaussian free field and random geometry, an important role is played by the so-called Liouville measure. This is a random measure which can formally be written as the exponential of the Gaussian free field. While the Gaussian free field is a random generalized function and exponentiating it is an operation one can not naively perform, there is a framework for making rigorous sense of such objects. This framework is known as Gaussian multiplicative chaos and is a type of renormalization procedure to define this exponential. The original ideas of the theory go back to Kahane [17], but we also refer the interested reader to the extensive review of Rhodes and Vargas [25] as well as the concise and elegant approach of Berestycki [4] for proving existence and uniqueness of the measure.

Thus motivated by the central limit theory of Rider and Virág, a natural question is whether multiplicative chaos measures can be constructed from the characteristic polynomials of the Ginibre ensemble and can the limiting measure be connected to these objects appearing in random geometry. Recently, multiplicative chaos measures have been constructed from characteristic polynomials of random matrices in the setting of random unitary and random Hermitian matrices – see [6, 20, 28]. What one would expect from these results is that $\frac{|\det(G_N - z)|^\gamma}{\mathbb{E} |\det(G_N - z)|^\gamma} d^2 z$ converges in law to a multiplicative chaos measure as $N \rightarrow \infty$. Moreover, a central question in [6, 20, 28] is to have precise asymptotics for quantities corresponding to $\mathbb{E} \prod_{j=1}^k |\det(G_N - z_j)|^{\gamma_j}$, so Theorem 1.1 is a first step in this direction as well.

1.2. Interpretation and comments about Theorem 1.1. We now make a few brief comments about Theorem 1.1. First of all, our restriction to $\gamma \in (0, 2)$ is merely for simplifying the proof – we expect that this result holds for $\gamma \in (-2, \infty)$ (or in fact for $\gamma \in \mathbb{C}$ with $\text{Re}(\gamma) > -2$), but to simplify our presentation, we focus on this case. From our point of view, the reason to restrict to $|z| < 1$ is that this is a more interesting case than $|z| > 1$: one should expect from [26], that for each $z \in \mathbb{C}$ for which $|z| > 1$, $\log |\det(G_N - z)| - \mathbb{E} \log |\det(G_N - z)|$ converges in law to a real valued Gaussian random variable, and instead of having $N^{\gamma^2/8}$ in Theorem 1.1, one should have a quantity converging as $N \rightarrow \infty$. We expect that this could be proven using a similar approach as the one we take here (using the results of [21] with $|z| > 1$), but we do not explore this further. Note that another reason to distinguish between $|z| < 1$ and $|z| > 1$ is that in our normalization, the unit disk is the support of the equilibrium measure for the Ginibre ensemble, so it is the set where the eigenvalues should accumulate in the large N limit.

We also point out that Theorem 1.1 is easy to justify on a heuristic level. Indeed, proving this result for $z = 0$ is very simple, as the relevant orthogonal polynomials can be calculated explicitly (see Lemma 2.2 for the definition and importance of the orthogonal polynomials). To heuristically justify our result for $z \neq 0$, we point out that from [26], one would might

expect that $\log |\det(G_N - z)| - \mathbb{E} \log |\det(G_N - z)|$ is a stationary stochastic process inside the unit disk (recall that formally this converged to a Gaussian process with translation invariant covariance), which would suggest that in Theorem 1.1, the only z -dependent contribution can come from $\mathbb{E} \log |\det(G_N - z)|$. Using e.g. [1, Theorem 2.1], one would expect that

$$\begin{aligned} \mathbb{E} \log |\det(G_N - z)| &= N \int_{|w|<1} \log |w - z| \frac{d^2 w}{\pi} + \frac{1}{8\pi} \int_{|w|<1} \Delta_w \log |w - z| d^2 w + o(1) \\ &= \frac{N}{2}(|z|^2 - 1) + \frac{1}{4} + o(1), \end{aligned}$$

which suggests that $\mathbb{E} |\det(G_N - z)|^\gamma = \mathbb{E} |\det(G_N)|^\gamma e^{\frac{\gamma}{2} N |z|^2} (1 + o(1))$. This is indeed true by Theorem 1.1.

Finally based on the analogy with the case of random Hermitian matrices from [18, 6] as well as the CLT of Rider and Virág [26] (and that from [1]), it would be natural to expect that a more general Fisher-Hartwig formula exists also for the Ginibre ensemble. We expect that the correct formulation would be the following: let z_j be distinct fixed points in the unit disk, $\gamma_j > 0$ for all $j = 1, \dots, k$, and $f : \mathbb{C} \rightarrow \mathbb{R}$ smooth enough with compact support in the unit disk (for simplicity), then

$$\begin{aligned} \mathbb{E} e^{\text{Tr} f(G_N)} \prod_{j=1}^k |\det(G_N - z_j)|^{\gamma_j} &= e^{N \int_{|z|<1} f(z) \frac{d^2 z}{\pi} + \frac{1}{8\pi} \int_{|z|<1} \Delta f(z) d^2 z + \frac{1}{4\pi} \int_{|z|<1} |\nabla f(z)|^2 d^2 z - \sum_{j=1}^k \gamma_j f(z_j)} \\ &\quad \times (1 + o(1)) \prod_{j=1}^k N^{\frac{\gamma_j^2}{8}} e^{\frac{N}{2} \gamma_j (|z_j|^2 - 1)} \frac{(2\pi)^{\frac{\gamma_j}{4}}}{G(1 + \frac{\gamma_j}{2})} \prod_{i<j} |z_i - z_j|^{-\frac{\gamma_i \gamma_j}{2}}. \end{aligned}$$

In fact, it's natural to expect that a related formula exists for more general (one-cut regular) ensembles. Unfortunately, we suspect that this kind of results with several singularities or non-zero f are out of reach with current tools.

1.3. Outline of the article. The outline of this article is the following. In Section 2, we recall how orthogonal polynomials, which are orthogonal with respect to the weight $F(w) = |w - z|^\gamma e^{-N|w|^2}$ (supported on the whole complex plane), are related to expectations of the form relevant to Theorem 1.1. We also recall a result of Balogh, Bertola, Lee, and McLaughlin which lets us transform orthogonality with respect to F into orthogonality with respect to a weight which is supported on a contour in \mathbb{C} . In Section 3, we recall how to encode these orthogonal polynomials associated to a contour into a Riemann-Hilbert problem, as well as generalize differential identities from [18, 10, 12] to facilitate efficient asymptotic analysis of the determinant of the moment matrix. Then in Section 4, we use results from [21] to solve our Riemann-Hilbert problem asymptotically. Finally in Section 5, we use our asymptotic solution of the Riemann-Hilbert problem to study the asymptotics of our differential identity, and prove Theorem 1.1 by integrating the differential identity. For completeness, we also recall some basic facts about orthogonal polynomials and Riemann-Hilbert problems as well as some of the results of [21] in appendices.

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2. THE GINIBRE ENSEMBLE AND ORTHOGONAL POLYNOMIALS

In this section, we recall some basic facts about the complex Ginibre ensemble, such as the distribution of the eigenvalues, how expectations of suitable functions of eigenvalues of Ginibre random matrices can be expressed in terms of determinants of complex moment matrices, as

well as how such questions relate to orthogonal polynomials. We also recall results from [3, 21] which show that the orthogonal polynomials associated to the expectation $\mathbb{E}|\det(G_N - z)|^\gamma$ also satisfy suitable orthogonality conditions on certain contours in the complex plane. Then in Section 3, we apply these results to transform the analysis of $\mathbb{E}|\det(G_N - z)|^\gamma$ into a question of the asymptotic analysis of a suitable Riemann-Hilbert problem. For the convenience of the reader, we sketch proofs of some of the statements of this section in Appendix A.

As stated in Theorem 1.1, G_N is a random $N \times N$ matrix whose entries are i.i.d. and distributed as $N^{-1/2}Z$, where Z is a standard complex Gaussian. We recall that the law of the eigenvalues of G_N can then be expressed in the following form [8]

$$(2.1) \quad \mathbb{P}(d^2 z_1, \dots, d^2 z_N) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \prod_{j=1}^N e^{-N|z_j|^2} d^2 z_j$$

on \mathbb{C}^N . Here the normalizing constant Z_N is

$$Z_N = \pi^N \frac{\prod_{k=1}^N k!}{N^{\frac{N(N+1)}{2}}}.$$

We'll denote integration with respect to $\mathbb{P}(d^2 z_1, \dots, d^2 z_N)$ by \mathbb{E} – so we suppress the dependence on N in our notation.

We now recall a Heine-Szegő-type identity which connects the Ginibre ensemble to determinants of complex moment matrices.

Lemma 2.1. *Let $F : \mathbb{C} \rightarrow \mathbb{R}$ be regular enough (so that the integrals below are well defined), then*

$$\mathbb{E} \prod_{j=1}^N F(z_j) = \frac{N!}{Z_N} D_{N-1}(F) := \frac{N!}{Z_N} \det \left(\int_{\mathbb{C}} w^i \overline{w}^j F(w) e^{-N|w|^2} d^2 w \right)_{i,j=0}^{N-1}.$$

This is a straightforward generalization of a corresponding identity for random Hermitian and random unitary matrices and relies on noticing that $\prod_{i < j} |z_i - z_j|^2$ in (2.1) can be written in terms of the Vandermonde determinant which then allows this determinantal representation. We omit further details.

The next fact we need is the connection between $D_{N-1}(F)$ defined in Lemma 2.1 and suitable orthogonal polynomials.

Lemma 2.2. *Let $F : \mathbb{C} \rightarrow [0, \infty)$ be regular enough (so that $\int_{\mathbb{C}} |w|^k F(w) e^{-N|w|^2} d^2 w < \infty$ for all $k \geq 0$) and Lebesgue almost everywhere positive. Then there exists a sequence of polynomials $(p_j)_{j=0}^\infty$ so that $p_j(w)$ is a degree j polynomial in w with the coefficient of w^j being positive and*

$$(2.2) \quad \int_{\mathbb{C}} p_j(w) \overline{p_k(w)} F(w) e^{-N|w|^2} d^2 w = \delta_{j,k}.$$

Moreover, if we write χ_j for the (positive) coefficient of w^j in $p_j(w)$, then

$$(2.3) \quad D_{N-1}(F) = \prod_{j=0}^{N-1} \chi_j^{-2}.$$

Note that p_j and χ_j depend on N as well as F . We suppress this in our notation. The proof of Lemma 2.2 is standard and relies on the determinantal form of the Gram-Schmidt procedure. In particular, one finds the following representation for the polynomials p_j :

$$(2.4) \quad p_j(w) = \frac{1}{\sqrt{D_{j-1}^{(N)}(F)D_j^{(N)}(F)}} \begin{vmatrix} \int_{\mathbb{C}} F(s)e^{-N|s|^2} d^2s & \cdots & \int_{\mathbb{C}} s^j F(s)e^{-N|s|^2} d^2s \\ \vdots & & \vdots \\ \int_{\mathbb{C}} \bar{s}^{j-1} F(s)e^{-N|s|^2} d^2s & \cdots & \int_{\mathbb{C}} \bar{s}^{j-1} s^j F(s)e^{-N|s|^2} d^2s \\ 1 & \cdots & w^j \end{vmatrix},$$

where we have used the notation $D_k^{(N)}(F) = \det(\int_{\mathbb{C}} s^i \bar{s}^j F(s)e^{-N|s|^2} ds)_{i,j=0}^k$ (so comparing to our previous notation, $D_{N-1}(F) = D_{N-1}^{(N)}(F)$). The statements of the lemma follow easily from this representation and we omit further details.

The next ingredient we shall need for our Riemann-Hilbert problem is a fact noticed in [3], namely that in the special case when $F(w) = |w - z|^\gamma$, the polynomials p_j from Lemma 2.2 satisfy certain orthogonality relations on suitable contours in the complex plane as well. To simplify notation slightly, we shall first note that the law of $(z_i)_{i=1}^N$ is invariant under rotations: it follows easily from (2.1) that for fixed $\theta \in \mathbb{R}$, $(e^{i\theta} z_j)_{j=1}^N$ has the same law as $(z_j)_{j=1}^N$. From this it follows that $\mathbb{E}|\det(G_N - z)|^\gamma = \mathbb{E}|\det(G_N - |z|)|^\gamma$. We thus see that for Theorem 1.1, it's enough to understand the asymptotics of $\mathbb{E}|\det(G_N - x)|^\gamma$ for $x \in (0, 1)$. To emphasize this we now restrict our attention to weights F that are relevant to this expectation, we fix our notation in the following definition.

Definition 2.3. For $x \in (0, 1)$ and $\gamma \in (0, 2)$, let $F : \mathbb{C} \rightarrow \mathbb{R}$, $F(w) = |w - x|^\gamma$. Moreover, let $(p_j)_{j=0}^\infty$ be the polynomials from Lemma 2.2 associated to this F and let χ_j be the coefficient of w^j in $p_j(w)$.

The statement about orthogonality on suitable contours discovered in [3, Lemma 3.1] is the following.

Lemma 2.4 (Balogh, Bertola, Lee, and McLaughlin). *Let Σ be a simple, smooth, and closed contour in the complex plane, and let it encircle $[0, x]$, possibly passing through x , but not other points of $[0, x]$, and let it be oriented in the counter-clockwise direction. Let*

$$(2.5) \quad f(w) = w^{-\frac{\gamma}{2}}(w - x)^{\frac{\gamma}{2}} e^{-N x w},$$

where the roots are according to the principal branch (so the branch cut of f is $[0, x]$). Then for $0 \leq k \leq j$,

$$(2.6) \quad \oint_{\Sigma} p_j(w) w^{-k} f(w) \frac{dw}{2\pi i w} = \begin{cases} 0, & k < j \\ \frac{1}{\pi} \frac{N^{1+\frac{\gamma}{2}+k}}{\Gamma(1+\frac{\gamma}{2}+k)} \frac{1}{\chi_j}, & k = j \end{cases}.$$

As the situation considered in [3] is slightly different – for them γ is proportional to N , and their result is stated for contours avoiding x , we sketch a proof in Appendix A. We also point out that if Σ were the unit circle, (2.6) would look like a basic orthogonality condition for polynomials on the unit circle. Thus (as in [3, 21]) it is fruitful to define a dual family of polynomials which are orthogonal to the polynomials p_j with respect to the pairing coming from (2.6). We now recall how these dual orthogonal polynomials are constructed and how their leading order coefficient is related to χ_j .

Lemma 2.5. *Let Σ and f be as in Lemma 2.4. Moreover, for $w \neq 0$ and $j \geq 0$, let*

$$(2.7) \quad q_j(w^{-1}) = \sqrt{\frac{\pi \Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2} + j + 1}}} \frac{1}{\sqrt{\widehat{D}_{j-1} \widehat{D}_j}} \begin{vmatrix} \oint_{\Sigma} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^{j-1} f(s) \frac{ds}{2\pi i s} & 1 \\ \vdots & & \vdots & \vdots \\ \oint_{\Sigma} s^{-j} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^{-1} f(s) \frac{ds}{2\pi i s} & w^{-j} \end{vmatrix},$$

where

$$(2.8) \quad \widehat{D}_j = \det \left(\oint_{\Sigma} w^{-(k-l)} f(w) \frac{dw}{2\pi i w} \right)_{k,l=0}^j.$$

These functions are well defined for all j and if we write $\widehat{\chi}_j$ for the coefficient of w^{-j} in $q_j(w^{-1})$, then for $k, j \geq 0$

$$(2.9) \quad \oint_{\Sigma} p_j(w) q_k(w^{-1}) f(w) \frac{dw}{2\pi i w} = \delta_{j,k}$$

and

$$(2.10) \quad \widehat{\chi}_j = \chi_j \frac{\pi \Gamma(1 + \frac{\gamma}{2} + j)}{N^{1 + \frac{\gamma}{2} + j}}.$$

Again, we offer a sketch of a proof in Appendix A, as such a result isn't formulated precisely in this form in [3, 21]. We now turn to the Riemann-Hilbert problem and the differential identity related to $D_{N-1}(F)$.

3. THE RIEMANN-HILBERT PROBLEM AND THE DIFFERENTIAL IDENTITY

We are now in a position to encode our polynomials into a Riemann-Hilbert problem in a similar way as in [3, 21] as well as state our differential identity. The proof of the differential identity is a modification of those appearing in [10, 12, 18], but as the differential identity in our case is slightly more complicated, we offer details for the proof in Appendix B.

We begin by defining the object that will satisfy a Riemann-Hilbert problem.

Definition 3.1. Let Σ be as in Lemma 2.4. For $w \notin \Sigma$ and $j \geq 1$, let

$$(3.1) \quad Y(w) = Y_j(w) = \begin{pmatrix} \frac{1}{\chi_j} p_j(w) & \frac{1}{\chi_j} \oint_{\Sigma} \frac{s^{-(j-1)} p_j(s) f(s)}{s-w} \frac{ds}{2\pi i s} \\ -\chi_{j-1} w^{j-1} q_{j-1}(w^{-1}) & -\chi_{j-1} \oint_{\Sigma} \frac{q_{j-1}(s^{-1}) f(s)}{s-w} \frac{ds}{2\pi i s} \end{pmatrix}.$$

Note that for each j , Y_j also depends on N , x , γ , as well as the contour Σ we have not yet fixed, but we suppress this in our notation.

As originally noticed by Fokas, Its, and Kitaev [15], such an object indeed satisfies a Riemann-Hilbert problem:

Lemma 3.2. For each $j \geq 1$, $Y = Y_j$ is the unique solution to the following Riemann-Hilbert problem.

- $Y : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- Y has continuous boundary values on Σ (denote by Y_+ the limit from the side of the origin and by Y_- the limit from the side of infinity) and they satisfy the following jump relation: for $w \in \Sigma$

$$(3.2) \quad Y_+(w) = Y_-(w) \begin{pmatrix} 1 & w^{-j} f(w) \\ 0 & 1 \end{pmatrix}.$$

- As $w \rightarrow \infty$,

$$(3.3) \quad Y(w) = (I + \mathcal{O}(w^{-1})) w^{j\sigma_3} = (I + \mathcal{O}(w^{-1})) \begin{pmatrix} w^j & 0 \\ 0 & w^{-j} \end{pmatrix}$$

where I is the 2×2 identity matrix and $\mathcal{O}(w^{-1})$ denotes a 2×2 matrix whose entries are bounded by $|w|^{-1}$ as $w \rightarrow \infty$.

The proof is the standard one – uniqueness of a solution follows from Liouville’s theorem, the jump conditions from the Sokhotski-Plemelj theorem, the asymptotic behavior from the orthogonality conditions, and the continuity of the boundary values follows from the fact that boundary values of the Cauchy transform associated to a smooth contour map Hölder continuous functions into Hölder continuous functions (see e.g. the discussion leading to [23, §19] for more about this fact). We omit further details of the proof and refer to e.g. [9, 19].

As we have seen in Lemma 2.2, to understand $D_{N-1}(F)$, we are a priori interested in χ_j for all $j \leq N-1$, which would suggest that one would need to solve the RHP for all j . Due to a differential identity we now describe, it’s enough for us to only solve the problem for Y_N and Y_{N+1} .

Lemma 3.3. *Let us write $D_{N-1}(F; \gamma) = D_{N-1}(F)$ (where again $F(w) = |w - x|^\gamma$). Let us also write κ_j for the coefficient of w^{j-1} in $p_j(w)$. Then for $\gamma > 0$*

$$\begin{aligned} & \partial_\gamma \log D_{N-1}(F; \gamma) \\ &= - \left(N + \frac{\gamma}{2} \right) \frac{\partial_\gamma \widehat{\chi}_N}{\widehat{\chi}_N} + \frac{\gamma}{2} x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} \oint_\Sigma p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} - Nx \frac{p_{N+1}(0)}{\chi_{N+1}} \frac{\partial_\gamma q_N(0)}{\widehat{\chi}_N} \\ & \quad - N \frac{\partial_\gamma \chi_N}{\chi_N} - \frac{\gamma}{2} \frac{\partial p_N(x)}{\partial \gamma} x \oint_\Sigma \frac{q_N(w^{-1}) f(w)}{w-x} \frac{dw}{2\pi i w} + Nx \left(\frac{\partial_\gamma \kappa_N}{\chi_N} - \frac{\partial_\gamma \chi_N}{\chi_N} \frac{\kappa_{N+1}}{\chi_{N+1}} \right) \\ & \quad + \partial_\gamma \sum_{j=0}^{N-1} \log \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2}}}. \end{aligned}$$

We give a proof in Appendix B. One can easily check that all of the quantities here can be expressed in terms of Y_N and Y_{N+1} – e.g. $\chi_N \widehat{\chi}_N = -Y_{N+1,21}(0)$, from which one can solve χ_N . See Section 5 for further details. We now move onto the asymptotic analysis of Y_N and Y_{N+1} by solving their RHPs.

4. SOLVING THE RIEMANN-HILBERT PROBLEM FOR Y_N ASYMPTOTICALLY

In this section we recall from [21] the asymptotic solution of the RHP for Y_N . In fact, we’ll consider a minor generalization of their situation where we study the asymptotics of Y_{N+k} , where k is a fixed integer – for our differential identity, we only need $k = 0$ and $k = 1$. Again we offer details of the argument in Appendix C since the question is slightly different from that in [21]. For intuition and further discussion concerning the approach, we refer to [21] and references therein.

As typical in this type of Riemann-Hilbert problems, using approximate problems which can be solved explicitly, we will transform this problem into a “small-norm” problem which can be solved asymptotically in terms of a Neumann-series. The solutions to the approximate problems are called parametrices, and we will need two of them: one close to the point x , and one far away from it. The one close to x is called the local parametrix and the one far from it is the global parametrix. We begin with a transformation that normalizes our problem at infinity and enables “opening lenses”, then we recall from [21] the global and local parametrices relevant to us. Finally we will consider the solution of the small norm problem.

4.1. Transforming the problem. The goal of the transformation procedure is to have a RHP which is normalized at infinity (the sought function converges to the identity matrix as $w \rightarrow \infty$) and for which the jump matrix is close to the identity as $N \rightarrow \infty$. This allows formulating the problem in terms of a certain singular integral equation which can be solved in terms of a suitable Neumann-series. We begin by normalizing the function at infinity. To do this, let us write $\text{Ext}(\Sigma)$ for the unbounded component of $\mathbb{C} \setminus \Sigma$ and $\text{Int}(\Sigma)$ for the bounded one (recall that we still have not fixed Σ , but we will do this shortly), and define

$$(4.1) \quad \ell = \log x - x^2 \quad \text{and} \quad g(w) = \begin{cases} \log w, & w \in \text{Ext}(\Sigma) \\ \ell + xw, & w \in \text{Int}(\Sigma) \end{cases}.$$

As we are only giving a brief overview of the approach of [21], we refer to [3, 21] for a discussion of why ℓ and g are chosen so. Throughout this section, we will be working with $Y = Y_{N+k}$ and we will drop for now the index $N+k$ from our notation. We then define

$$(4.2) \quad T(w) = e^{-(N+k)\frac{\ell}{2}\sigma_3} Y(w) e^{-(N+k)g(w)\sigma_3} e^{(N+k)\frac{\ell}{2}\sigma_3}.$$

Note that from the asymptotic behavior of Y , namely (3.3), and our choice of g in $\text{Ext}(\Sigma)$, we see that $T(w) = I + \mathcal{O}(w^{-1})$ as $w \rightarrow \infty$.

Let us next fix the contour Σ . Let

$$(4.3) \quad \begin{aligned} \Sigma &= \{w \in \mathbb{C} : \text{Re}(xw + \ell - \log w) = 0, \text{Re}(w) \leq x\} \\ &= \{u + iv \in \mathbb{C} : u^2 + v^2 = x^2 e^{2x(u-x)}, u \leq x\}. \end{aligned}$$

The point of choosing our jump contour to be this one will be evident shortly as we'll perform another transformation which will result in a jump matrix close to the identity when off of our contour Σ . Before going into our next transformation, we point out the following fact (see also [21, Lemma 4]).

Lemma 4.1. *For each $x \in (0, 1)$, Σ is a smooth, simple closed loop inside the unit disk. It encircles $[0, x]$ (passing through x , but not other points), and*

$$\text{Re}(xw + \ell - \log w) \begin{cases} > 0, & w \in \text{Int}(\Sigma) \\ < 0, & w \in \{s \in \text{Ext}(\Sigma) : |s| \leq 1\} \end{cases},$$

moreover $\sup_{|w|=1} \text{Re}(xw + \ell - \log w) < 0$ for all $x \in (0, 1)$.

Note that in particular, Σ satisfies the conditions of Lemma 2.4. The proof is given in Appendix C.

Our next transformation allows us to perform a Deift-Zhou non-linear steepest descent-type argument by opening lenses. Our lens will now essentially be the unit circle combined with the interval $[0, x]$. We define

$$(4.4) \quad S(w) = \begin{cases} T(w), & |w| > 1 \\ T(w) \begin{pmatrix} 1 & 0 \\ w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix}, & w \in \{s \in \text{Ext}(\Sigma) : |s| < 1\} \\ T(w) \begin{pmatrix} 1 & 0 \\ -w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{-(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix}, & w \in \text{Int}(\Sigma) \setminus [0, x] \end{cases},$$

where as before, the roots are according to the principal branch.

We now describe the RHP satisfied by this function.

Lemma 4.2. *Let $C = [0, x] \cup \Sigma \cup \{w \in \mathbb{C} : |w| = 1\}$. Orient $[0, x]$ from 0 to x so that the + side of the interval is the upper half plane. Orient the unit circle so that the inside of the circle is the + side of the contour, and orient Σ in the counter-clockwise direction (i.e. we let the + side of the contour be the side of the origin and the - side of the contour be the side of infinity). Then S satisfies the following Riemann-Hilbert problem.*

- $S : \mathbb{C} \setminus C \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.

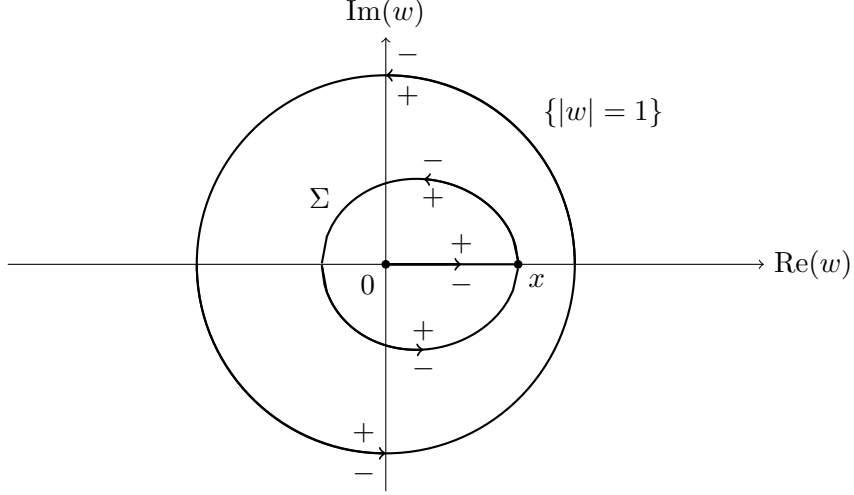


FIGURE 1. S -RHP and the opening of lenses.

- S has continuous boundary values on $C \setminus \{x\}$, and these satisfy the following jump conditions:

$$(4.5) \quad S_+(w) = S_-(w) \begin{pmatrix} 1 & 0 \\ w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix}, \quad |w| = 1,$$

$$(4.6) \quad S_+(w) = S_-(w) \begin{pmatrix} 0 & e^{kxw}(w-x)^{\frac{\gamma}{2}} w^{-\frac{\gamma}{2}} \\ -e^{-kxw}(w-x)^{-\frac{\gamma}{2}} w^{\frac{\gamma}{2}} & 0 \end{pmatrix}, \quad w \in \Sigma \setminus \{x\},$$

and

$$(4.7) \quad S_+(w) = S_-(w) \begin{pmatrix} 1 & 0 \\ 2i \sin \frac{\pi\gamma}{2} |w|^{\gamma/2} |w-x|^{-\frac{\gamma}{2}} e^{-kxw} e^{-(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix}, \quad w \in (0, x).$$

- As $w \rightarrow 0$ $S(w)$ is bounded (actually $S(0)$ exists) and as $w \rightarrow x$ (off of C),

$$(4.8) \quad S(w) = \begin{pmatrix} \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & \mathcal{O}(1) \\ \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & \mathcal{O}(1) \end{pmatrix}.$$

- As $w \rightarrow \infty$, $S(w) = I + \mathcal{O}(w^{-1})$.

The proof is in Appendix C.

Our next task is to find the approximate solutions. The first one corresponds to focusing on a problem where we only consider the jump condition (4.6) (the global parametrix), while the second one approximates the RHP close to the point x (the local parametrix) as well as approximately matches the global solution on the boundary of a small neighborhood of the point x .

4.2. The global parametrix. Here we first look for a function $P^{(\infty)} : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2}$ which satisfies the jump condition (4.6) and is normalized at infinity. We simply mention that one can easily check that the function

$$(4.9) \quad P^{(\infty)}(w) = \begin{cases} \begin{pmatrix} w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} & 0 \\ 0 & w^{-\frac{\gamma}{2}}(w-x)^{\frac{\gamma}{2}} \end{pmatrix}, & w \in \text{Ext}(\Sigma) \\ \begin{pmatrix} 0 & e^{kxw} \\ -e^{-kxw} & 0 \end{pmatrix}, & w \in \text{Int}(\Sigma) \end{cases}$$

satisfies these conditions.

If we were to take this as our global parametrix, we would obtain a small norm problem for $\gamma < 2$ and it could be solved as an expansion in $N^{\frac{\gamma}{2}-1}$. This is not quite sufficient for us – we have terms proportional to N in our differential identity, so we would like an expansion in N^{-1} . What turns out to be the correct thing to do is to take the following function as the global parametrix:

$$(4.10) \quad \widehat{P}^{(\infty)}(w) = \begin{pmatrix} 1 & (N+k)^{\frac{\gamma}{2}-1} \frac{x e^{kx^2(1-x^2)^{\frac{\gamma}{2}-1}}}{\Gamma(\frac{\gamma}{2})} (w-x)^{-1} \\ 0 & 1 \end{pmatrix} P^{(\infty)}(w).$$

Note that this is still analytic off of Σ , has continuous boundary values on $\Sigma \setminus \{x\}$, and satisfies the jump condition (4.6). Moreover, $\widehat{P}^{(\infty)}$ is normalized at infinity. We also mention that in the $\gamma > 2$ case, this function would need to be adjusted slightly (increasingly much for large γ) – see Remark 4.3 as well as [21].

4.3. The local parametrix. Here we look for a function which has the same jump conditions as S in a small enough neighborhood of x and up to a term of order $\mathcal{O}(N^{-1})$, agrees with $\widehat{P}^{(\infty)}$ on the boundary of this neighborhood. To do this, let U be a small but fixed circular neighborhood of x . We assume that the neighborhood is small enough so that $0, 1 \notin U$. We also define a coordinate change $\zeta : U \rightarrow \mathbb{C}$,

$$(4.11) \quad \zeta(w) = -(N+k)(xw - \log w + \ell) = \frac{N+k}{x}(1-x^2)(w-x) + \mathcal{O}((N+k)(w-x)^2)$$

where the branch of the logarithm is the principal one. ζ blows up U conformally into a large neighborhood of the origin. From the definition of Σ , it follows that ζ maps $U \cap \Sigma$ into a segment of the imaginary axis. We also assume that U is so small that ζ is one-to-one. We then define our local parametrix in the following way: for $z \in U$, let

$$(4.12) \quad P^{(x)}(w) = \begin{pmatrix} 1 & Q(w) \\ 0 & 1 \end{pmatrix} P^{(\infty)}(w),$$

where

$$(4.13) \quad Q(w) = w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{kxw} \frac{\Gamma(\frac{\gamma}{2}, \zeta(w)) e^{\zeta(w)}}{\Gamma(\frac{\gamma}{2})} + (N+k)^{\frac{\gamma}{2}-1} \frac{x e^{kx^2(1-x^2)^{\frac{\gamma}{2}-1}}}{\Gamma(\frac{\gamma}{2})(w-x)} \\ - \left(\frac{w\zeta(w)}{w-x} \right)^{\frac{\gamma}{2}} \frac{e^{kxw}}{\zeta(w)\Gamma(\frac{\gamma}{2})}$$

and $\Gamma(\nu, \zeta)$ is the upper incomplete gamma-function, which we can write in the following way

$$\Gamma(\nu, \zeta) = \Gamma(\nu) (1 - \zeta^\nu \gamma^*(\nu, \zeta)),$$

where $\gamma^*(\nu, \zeta) = e^{-\zeta} \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(j+\nu+1)}$ is an entire function of ζ .

Remark 4.3. Note that the local parametrix can also be written in the form

$$P^{(x)}(w) = \begin{pmatrix} 1 & w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} \zeta(w)^{\frac{\gamma}{2}} e^{kxw} \frac{1}{\Gamma(\frac{\gamma}{2})} \left[\zeta(w)^{-\frac{\gamma}{2}} e^{\zeta(w)} \Gamma(\frac{\gamma}{2}, \zeta(w)) - \frac{1}{\zeta(w)} \right] \\ 0 & 1 \end{pmatrix} \widehat{P}^{(\infty)}(w).$$

This representation is useful for checking the matching condition since from the asymptotic expansion of the incomplete gamma function (see (C.1)), one has

$$\zeta^{-\frac{\gamma}{2}} e^{\zeta} \Gamma\left(\frac{\gamma}{2}, \zeta\right) - \frac{1}{\zeta} = \mathcal{O}(\zeta^{-2})$$

for large $|\zeta|$. In fact, from this representation one can easily deduce what the local and global parametrix must be for larger γ : one simply subtracts from $\zeta^{-\frac{\gamma}{2}} e^{\zeta} \Gamma(\frac{\gamma}{2}, \zeta)$ higher and higher order terms from the asymptotic expansion and then the global parametrix is chosen to cancel the poles coming from these correction terms from the asymptotic expansion. The cancellation of the poles is important for the parametrix to have the same asymptotic behavior at x as S does. Again, for further details, see [21].

We can now state the RHP for $P^{(x)}$.

Lemma 4.4. $P^{(x)}$ satisfies the following Riemann-Hilbert problem.

- $P^{(x)} : U \setminus ([0, x] \cup \Sigma) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- $P^{(x)}$ has continuous boundary values on $U \cap ([0, x] \cup \Sigma) \setminus \{x\}$ and they satisfy the following jump conditions:

$$(4.14) \quad P_+^{(x)}(w) = P_-^{(x)}(w) \begin{pmatrix} 0 & (w-x)^{\frac{\gamma}{2}} w^{-\frac{\gamma}{2}} e^{kxw} \\ -(w-x)^{-\frac{\gamma}{2}} w^{\frac{\gamma}{2}} e^{-kxw} & 0 \end{pmatrix}, \quad w \in \Sigma \setminus \{x\}$$

and

$$(4.15) \quad P_+^{(x)}(w) = P_-^{(x)}(w) \begin{pmatrix} 1 & 0 \\ 2i \sin \frac{\pi\gamma}{2} |w|^{\gamma/2} |w-x|^{-\frac{\gamma}{2}} e^{-kxw} e^{-(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix}, \quad w \in (0, x).$$

- As $w \rightarrow x$ (off of $\Sigma \cup [0, x]$),

$$(4.16) \quad P^{(x)}(w) = \begin{cases} \begin{pmatrix} \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & 0 \\ 0 & \mathcal{O}(|w-x|^{\frac{\gamma}{2}}) \end{pmatrix}, & w \in \text{Ext}(\Sigma) \\ \begin{pmatrix} \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & e^{kxw} \\ -e^{-kxw} & 0 \end{pmatrix}, & w \in \text{Int}(\Sigma) \end{cases}.$$

- We have for each fixed $k \in \mathbb{Z}$

$$(4.17) \quad P^{(x)}(w) \left[\widehat{P}^{(\infty)}(w) \right]^{-1} = I + \mathcal{O}(N^{-2+\frac{\gamma}{2}}),$$

uniformly in $w \in \partial U$ and $\gamma \in (0, 2)$.

Again for the proof, see Appendix C.

We are now in a position to perform our final transformation and complete our asymptotic analysis of Y_{N+k} .

4.4. The final transformation and asymptotic analysis. Our final transformation of the problem is the following one:

$$(4.18) \quad R(w) = \begin{cases} S(w) [P^{(x)}(w)]^{-1}, & w \in U \\ S(w) [\widehat{P}^{(\infty)}(w)]^{-1}, & w \in \mathbb{C} \setminus \overline{U} \end{cases}$$

and we now describe the RHP it satisfies.

Lemma 4.5. R is the unique solution to the following RHP:

- $R : \mathbb{C} \setminus (\partial U \cup \{|w| = 1\}) \rightarrow \mathbb{C}^{2 \times 2}$ is analytic.
- R has continuous boundary values on $\partial U \cup \{|w| = 1\}$ and these satisfy

$$(4.19) \quad R_+(w) = R_-(w) P^{(x)}(w) \left[\widehat{P}^{(\infty)}(w) \right]^{-1}, \quad w \in \partial U$$

and

(4.20)

$$R_+(w) = R_-(w) \widehat{P}^{(\infty)}(w) \begin{pmatrix} 1 & 0 \\ w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix} \left[\widehat{P}^{(\infty)}(w) \right]^{-1}, \quad |w| = 1.$$

- As $w \rightarrow \infty$, $R(w) = I + \mathcal{O}(w^{-1})$.

Moreover, if we write $I + \Delta_R$ for the jump matrix of R , then for each fixed $k \in \mathbb{Z}$, as $N \rightarrow \infty$, $\sup_{w \in \partial U} |\Delta_R(w)| = \mathcal{O}(N^{-2+\frac{\gamma}{2}})$ and $\sup_{|w|=1} |\Delta_R(w)| = \mathcal{O}(e^{-cN})$ for some $c > 0$.

Again, the proof is in Appendix C.

Now this RHP is one that's normalized at infinity and whose jump matrix is close to the identity when $N \rightarrow \infty$. Thus it can be solved asymptotically through the standard machinery. The outcome is the following estimate.

Lemma 4.6. *Let $\Gamma_R = \partial U \cup \{|w| = 1\}$ be the jump contour of R . Then*

$$(4.21) \quad R(w) = I + \mathcal{O}(N^{-2+\frac{\gamma}{2}}), \quad \lim_{w \rightarrow \infty} w[R(w) - I] = \mathcal{O}(N^{-2+\frac{\gamma}{2}})$$

uniformly in $w \in \mathbb{C} \setminus \Gamma_R$, $\gamma \in [0, \gamma_0]$ for any $\gamma_0 \in (0, 2)$, and for x in a compact subset of $(0, 1)$. Moreover, for any fixed $\epsilon > 0$

$$(4.22) \quad \partial_\gamma R(w) = \mathcal{O}(N^{-2+\frac{\gamma}{2}+\epsilon}) \quad \partial_\gamma \left[\lim_{w \rightarrow \infty} w(R(w) - I) \right] = \mathcal{O}(N^{-2+\frac{\gamma}{2}+\epsilon}),$$

uniformly in $w \in \mathbb{C} \setminus \Gamma_R$, $\gamma \in [0, \gamma_0]$ for any $\gamma_0 \in (0, 2)$ as well as uniformly in x when restricted to a compact subset of $(0, 1)$.

Armed with these estimates, we now turn to studying the asymptotic behavior of the differential identity and proving Theorem 1.1.

5. PROOF OF THEOREM 1.1: INTEGRATING THE DIFFERENTIAL IDENTITY

We summarize the asymptotics of our differential identity in the following lemma.

Lemma 5.1. *We have*

$$(5.1) \quad \partial_\gamma \log D_{N-1}(F; \gamma) = \frac{Nx^2}{2} + \partial_\gamma \sum_{j=0}^{N-1} \log \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2}}} + o(1)$$

where the $o(1)$ error is uniform in $\gamma \in [0, \gamma_0]$, $\gamma_0 \in (0, 2)$, and x in compact subsets of $(0, 1)$.

Proof. Fix $\epsilon > 0$ small. We start with the terms that require the evaluation of the Y -RHP at $w = 0 \in \text{Int}(\Sigma)$. In particular, we will first consider the logarithmic derivatives of χ and $\widehat{\chi}$. We begin by noting that $g(0) = \ell$ and the global parametrix is given by

$$\widehat{P}_{N+k}^{(\infty)}(0) = \begin{pmatrix} (N+k)^{\frac{\gamma}{2}-1} \frac{e^{kx^2} (1-x^2)^{\frac{\gamma}{2}-1}}{\Gamma(\frac{\gamma}{2})} & 1 \\ -1 & 0 \end{pmatrix}.$$

Let us look at the leading coefficients of our orthogonal polynomials: for each $k \in \mathbb{N}$ we have

$$\chi_{N+k} \widehat{\chi}_{N+k} = -Y_{N+k+1,21}(0), \quad \widehat{\chi}_{N+k} = \chi_{N+k} \frac{\pi \Gamma(1 + \frac{\gamma}{2} + N + k)}{N^{1+\frac{\gamma}{2}+N+k}}.$$

Applying the global parametrix and the error estimate (4.21), we have

$$Y_{N+k,21}(0) = T_{N+k,21}(0) = \left[R_{N+k}(0) \widehat{P}_{N+k}^{(\infty)}(0) \right]_{21} = -1 + \mathcal{O}\left(N^{-2+\frac{\gamma}{2}}\right).$$

Therefore

$$(5.2) \quad \chi_{N+k} = \left(\frac{\pi \Gamma \left(1 + \frac{\gamma}{2} + N + k \right)}{N^{1+\frac{\gamma}{2}+N+k}} \right)^{-\frac{1}{2}} \left(1 + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}} \right) \right)$$

and

$$(5.3) \quad \widehat{\chi}_{N+k} = \left(\frac{\pi \Gamma \left(1 + \frac{\gamma}{2} + N + k \right)}{N^{1+\frac{\gamma}{2}+N+k}} \right)^{\frac{1}{2}} \left(1 + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}} \right) \right).$$

For the logarithmic derivatives, we see from (4.22) that for any $\epsilon > 0$,

$$\partial_\gamma Y_{N+1,21}(0) = \partial_\gamma \left[R_{N+k}(0) \widehat{P}_{N+k}^{(\infty)}(0) \right]_{21} = \mathcal{O} \left(N^{-2+\frac{\gamma}{2}+\epsilon} \right).$$

Combining this with the standard asymptotics of the digamma function (which follows from Binet's second formula for the log-Gamma function, see e.g. [29, Section 12.32])

$$\frac{\Gamma'(u)}{\Gamma(u)} = \log u - \frac{1}{2u} + \mathcal{O}(u^{-2}), \quad u \rightarrow \infty,$$

we obtain

$$(5.4) \quad \begin{aligned} \frac{\partial_\gamma \chi_N}{\chi_N} &= -\frac{1}{2} \partial_\gamma \log \frac{\Gamma \left(1 + \frac{\gamma}{2} + N \right)}{N^{\frac{\gamma}{2}}} + \frac{1}{2} \frac{\partial_\gamma Y_{N+1,21}(0)}{Y_{N+1,21}(0)} \\ &= -\frac{\gamma+1}{8N} + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}+\epsilon} \right) \end{aligned}$$

$$(5.5) \quad \frac{\partial_\gamma \widehat{\chi}_N}{\widehat{\chi}_N} = \frac{\gamma+1}{8N} + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}+\epsilon} \right).$$

In particular,

$$(5.6) \quad - \left(N + \frac{\gamma}{2} \right) \frac{\partial_\gamma \widehat{\chi}_N}{\widehat{\chi}_N} - N \frac{\partial_\gamma \chi_N}{\chi_N} = \mathcal{O} \left(N^{-1+\frac{\gamma}{2}+\epsilon} \right).$$

Note that these estimates are all uniform in $0 \leq \gamma \leq \gamma_0 < 2$ and if we choose ϵ small enough, $\mathcal{O} \left(N^{-1+\frac{\gamma}{2}+\epsilon} \right) = o(1)$ uniformly in everything relevant.

We now consider the $p_{N+1}(0)\partial_\gamma q_N(0)$ -term. First of all

$$(5.7) \quad \begin{aligned} \frac{p_{N+1}(0)}{\chi_{N+1}} &= Y_{N+1,11}(0) \\ &= e^{(N+1)\ell} \left[R_{N+1}(0) \widehat{P}_{N+1}^{(\infty)}(0) \right]_{11} \\ &= e^{(N+1)\ell} (N+1)^{\frac{\gamma}{2}-1} \frac{e^{x^2} (1-x^2)^{\frac{\gamma}{2}-1}}{\Gamma \left(\frac{\gamma}{2} \right)} \left(1 + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}} \right) \right). \end{aligned}$$

Next, we need to evaluate Y at ∞ , which requires the global parametrix for $w \in \text{Ext}(\Sigma)$:

$$\widehat{P}_{N+k}^{(\infty)}(w) = \begin{pmatrix} w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} & w^{-\frac{\gamma}{2}}(w-x)^{\frac{\gamma}{2}}(N+k)^{\frac{\gamma}{2}-1} \frac{x e^{-kx^2} (1-x^2)^{\frac{\gamma}{2}-1}}{\Gamma \left(\frac{\gamma}{2} \right)} (w-x)^{-1} \\ 0 & w^{-\frac{\gamma}{2}}(w-x)^{\frac{\gamma}{2}} \end{pmatrix}.$$

Using the R -RHP estimate (4.21) one has

$$\begin{aligned}
q_N(0) &= -\frac{1}{\chi_N} \lim_{w \rightarrow \infty} w^{-N} Y_{N+1,21}(w) \\
&= -\frac{1}{\chi_N} e^{-(N+1)\ell} \lim_{w \rightarrow \infty} w \left[R_{N+1}(w) \widehat{P}_{N+1}^{(\infty)}(w) \right]_{21} \\
&= -\frac{1}{\chi_N} e^{-(N+1)\ell} \lim_{w \rightarrow \infty} w R_{N+1,21}(w) \\
&= -\frac{1}{\chi_N} e^{-(N+1)\ell} \mathcal{O} \left(N^{-2+\frac{\gamma}{2}} \right).
\end{aligned}$$

Combining this with (4.22) and (5.4), we see that

$$\begin{aligned}
\partial_\gamma q_N(0) &= -\frac{\partial_\gamma \chi_N}{\chi_N} q_N(0) - \frac{1}{\chi_N} e^{-(N+1)\ell} \lim_{w \rightarrow \infty} w \partial_\gamma R_{N+1,21}(w) \\
&= -\frac{1}{\chi_N} e^{-(N+1)\ell} \mathcal{O} \left(N^{-2+\frac{\gamma}{2}+\epsilon} \right).
\end{aligned}$$

Finally combining this with (5.7), (5.2), and (5.3) yields the asymptotics of the relevant term:

$$(5.8) \quad -Nx \frac{p_{N+1}(0)}{\chi_{N+1}} \frac{\partial_\gamma q_N(0)}{\widehat{\chi}_N} = \mathcal{O} \left(N^{-2+\gamma+\epsilon} \right),$$

which again under our assumptions is $o(1)$ uniformly in everything relevant.

We now move onto the κ -terms: we find by the definition of κ and Y (along with (4.21)) that

$$\begin{aligned}
\frac{\kappa_{N+k}}{\chi_{N+k}} &= \lim_{w \rightarrow \infty} w^{-N-k+1} (Y_{N+k,11}(w) - w^{N+k}) \\
&= \lim_{w \rightarrow \infty} w (T_{N+k,11}(w) - 1) \\
&= \lim_{w \rightarrow \infty} w \left(\left[R_{N+k}(w) \widehat{P}_{N+k}^{(\infty)}(w) \right]_{11} - 1 \right) \\
&= \frac{\gamma}{2} x + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}} \right).
\end{aligned}$$

Similarly from (4.22), we see that

$$\begin{aligned}
\frac{\partial_\gamma \kappa_{N+k}}{\chi_{N+k}} &= \partial_\gamma \frac{\kappa_{N+k}}{\chi_{N+k}} + \frac{\kappa_{N+k}}{\chi_{N+k}} \frac{\partial_\gamma \chi_{N+k}}{\chi_{N+k}} \\
&= \partial_\gamma \lim_{w \rightarrow \infty} w \left(\left[R_{N+k}(w) \widehat{P}_{N+k}^{(\infty)}(w) \right]_{11} - 1 \right) + \frac{\kappa_{N+k}}{\chi_{N+k}} \frac{\partial_\gamma \chi_{N+k}}{\chi_{N+k}} \\
&= \frac{1}{2} x + \frac{\kappa_{N+k}}{\chi_{N+k}} \frac{\partial_\gamma \chi_{N+k}}{\chi_{N+k}} + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}+\epsilon} \right).
\end{aligned}$$

Combining the last two estimates with (5.4), we find

$$\begin{aligned}
(5.9) \quad Nx \left(\frac{\partial_\gamma \kappa_N}{\chi_N} - \frac{\partial_\gamma \chi_N}{\chi_N} \frac{\kappa_{N+1}}{\chi_{N+1}} \right) &= Nx \left(\frac{x}{2} + \frac{\partial_\gamma \chi_N}{\chi_N} \left(\frac{\kappa_N}{\chi_N} - \frac{\kappa_{N+1}}{\chi_{N+1}} \right) + \mathcal{O} \left(N^{-2+\frac{\gamma}{2}+\epsilon} \right) \right) \\
&= \frac{Nx^2}{2} + \mathcal{O} \left(N^{-1+\frac{\gamma}{2}+\epsilon} \right).
\end{aligned}$$

The remaining terms in the differential identity (those involving the Cauchy-transforms) require the Y -RHP at the singularity $w = x$ and hence the local parametrix. For $w \in \text{Int}(\Sigma) \setminus$

$[0, x]$,

$$Y_{N+k}(w) = e^{(N+k)\frac{\ell}{2}\sigma_3} R_{N+k}(w) P_{N+k}^{(x)}(w) \begin{pmatrix} 1 & 0 \\ w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{-(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix} \\ \times e^{(N+k)xw\sigma_3} e^{(N+k)\frac{\ell}{2}\sigma_3}.$$

A straightforward computation shows that

$$P_{N+k}^{(x)}(w) \begin{pmatrix} 1 & 0 \\ w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{-(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix} = \begin{pmatrix} \tilde{P}_{N+k}(w) & e^{kxw} \\ -e^{-kxw} & 0 \end{pmatrix}$$

where

$$\tilde{P}_{N+k}(w) = w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{\zeta(w)} \left(1 - \frac{\Gamma(\frac{\gamma}{2}, \zeta(w))}{\Gamma(\frac{\gamma}{2})} \right) \\ - e^{kx(x-w)} (N+k)^{\frac{\gamma}{2}-1} \frac{x(1-x^2)^{\frac{\gamma}{2}-1}}{\Gamma(\frac{\gamma}{2})(w-x)} + \left(\frac{w\zeta(w)}{w-x} \right)^{\frac{\gamma}{2}} \frac{1}{\zeta(w)\Gamma(\frac{\gamma}{2})} \\ \xrightarrow{w \rightarrow x} \frac{(N+k)^{\frac{\gamma}{2}}(1-x^2)^{\frac{\gamma}{2}}}{\Gamma(1+\frac{\gamma}{2})} + \mathcal{O}\left(N^{\frac{\gamma}{2}-1}\right).$$

Indeed, one can check that the residual term $\mathcal{O}\left(N^{\frac{\gamma}{2}-1}\right)$ above is precisely given by

$$\frac{(N+k)^{\frac{\gamma}{2}-1}}{\Gamma(\frac{\gamma}{2})} x(1-x^2)^{\frac{\gamma}{2}-1} \left[\frac{1}{x} \frac{\gamma}{2} + \frac{1}{2x} \frac{1}{1-x^2} \left(1 - \frac{\gamma}{2} \right) + kx \right].$$

Letting $w \rightarrow x$ from $\text{Int}(\Sigma) \setminus [0, x]$, we get (from the above bound on \tilde{P} as well as (4.21))

$$Y_{N+k}(x) = \begin{pmatrix} x^{N+k} \left[R_{N+k,11}(x) \tilde{P}_{N+k}(x) - R_{N+k,12}(x) e^{-kx^2} \right] & e^{-Nx^2} R_{N+k,11}(x) \\ e^{(N+k)x^2} \left[R_{N+k,21}(x) \tilde{P}_{N+k}(x) - R_{N+k,22}(x) e^{-kx^2} \right] & x^{-(N+k)} R_{N+k,21} e^{kx^2} \end{pmatrix} \\ = \begin{pmatrix} x^{N+k} \frac{(N+k)^{\frac{\gamma}{2}}(1-x^2)^{\frac{\gamma}{2}}}{\Gamma(1+\frac{\gamma}{2})} [1 + \mathcal{O}(N^{-1})] & e^{-Nx^2} \left(1 + \mathcal{O}\left(N^{-2+\frac{\gamma}{2}}\right) \right) \\ -e^{Nx^2} [1 + \mathcal{O}(N^{-2+\gamma})] & x^{-(N+k)} e^{kx^2} \mathcal{O}\left(N^{-2+\frac{\gamma}{2}}\right) \end{pmatrix}.$$

We immediately see that

$$(5.10) \quad \oint_{\Sigma} p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} = \chi_N Y_{N,12}(x) = \chi_N e^{-Nx^2} \left(1 + \mathcal{O}\left(N^{-2+\frac{\gamma}{2}}\right) \right).$$

A similar argument (using (4.22), looking that the asymptotics of $\partial_{\gamma} \tilde{P}$, and (5.4)) shows that

$$(5.11) \quad x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} = -\frac{\partial}{\partial \gamma} \left[\frac{1}{\chi_N} Y_{N+1,21}(x) \right] = -\frac{1}{\chi_N} \partial_{\gamma} Y_{N+1,21}(x) + \frac{\partial_{\gamma} \chi_N}{\chi_N} \frac{1}{\chi_N} Y_{N+1,21}(x) \\ = \frac{e^{Nx^2}}{\chi_N} \mathcal{O}(N^{-2+\gamma+\epsilon}).$$

Combining with (5.10), we find (again with the required uniformity)

$$(5.12) \quad \frac{\gamma}{2} x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} \oint_{\Sigma} p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} = \mathcal{O}(N^{-2+\gamma+\epsilon}),$$

Similarly, we have

$$\oint_{\Sigma} \frac{q_N(w^{-1}) f(w)}{w-x} \frac{dw}{2\pi i w} = -\frac{1}{\chi_N} Y_{N+1,22}(x) = -\frac{1}{\chi_N} x^{-(N+1)} \mathcal{O}(N^{-2+\frac{\gamma}{2}})$$

and

$$\frac{\partial p_N(x)}{\partial \gamma} = \frac{\partial}{\partial \gamma} [\chi_N Y_{N,11}(x)] = \chi_N \partial_{\gamma} Y_{N,11}(x) + \frac{\partial \chi_N}{\chi_N} \chi_N Y_{N,11}(x) = \chi_N x^{N+1} \mathcal{O}(N^{\frac{\gamma}{2}+\epsilon}).$$

Combining these estimates, we conclude that (uniformly in everything relevant)

$$(5.13) \quad -\frac{\gamma}{2} \frac{\partial p_N(x)}{\partial \gamma} \oint_{\Sigma} \frac{q_N(w^{-1})f(w)}{w-x} \frac{dw}{2\pi i w} = \mathcal{O}(N^{-2+\gamma+\epsilon}).$$

Finally our lemma follows by substituting (5.6), (5.8), (5.9), (5.12) and (5.13) into the differential identity in Lemma 3.3. As mentioned, the $o(1)$ error is uniform in $\gamma \in [0, \gamma_0]$ as one can pick $\epsilon > 0$ such that $-1 + \frac{\gamma_0}{2} + \epsilon < 0$. The uniformity in x follows from the corresponding uniformity in x in our asymptotic estimates for R . \square

Proof of Theorem 1.1. Since the error term in (5.1) is uniform in $\gamma \in [0, \gamma_0]$, we can integrate both sides of (5.1) and obtain

$$\log \frac{D_{N-1}(F; \gamma)}{D_{N-1}(F; 0)} = \frac{N\gamma}{2} x^2 + \sum_{j=0}^{N-1} \log \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2}} \Gamma(j + 1)} + o(1).$$

Given that $G(u+1) = \Gamma(u)G(u)$ and $G(1) = 1$, we see that

$$\sum_{j=0}^{N-1} \log \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2}} \Gamma(j + 1)} = \log \frac{G(\frac{\gamma}{2} + N + 1)}{G(1 + \frac{\gamma}{2})G(N + 1)} - \frac{N\gamma}{2} \log N.$$

Let us recall the asymptotics for the logarithm of Barnes G-function (see e.g. [14, Theorem 1 and Theorem 2]):

$$\log G(u+1) = \frac{1}{12} - \log A + \frac{u}{2} \log 2\pi + \left(\frac{u^2}{2} - \frac{1}{12} \right) \log u - \frac{3u^2}{4} + O(u^{-2})$$

where A is the Glaisher-Kinkelin constant. In particular

$$\begin{aligned} \log \frac{G(\frac{\gamma}{2} + N + 1)}{G(N + 1)} &= \frac{\gamma}{4} \log 2\pi + \left(\frac{(N + \frac{\gamma}{2})^2}{2} - \frac{N^2}{2} \right) \log N \\ &\quad + \left[\frac{(N + \frac{\gamma}{2})^2}{2} - \frac{1}{12} \right] \log \left(1 + \frac{\gamma}{2N} \right) - \frac{3}{4} \left[(N + \frac{\gamma}{2})^2 - N^2 \right] \\ &= \frac{\gamma}{4} \log 2\pi + \left[\frac{N\gamma}{2} + \frac{\gamma^2}{8} \right] \log N - \frac{N\gamma}{2} + o(1). \end{aligned}$$

Therefore

$$\log \frac{D_{N-1}(F; \gamma)}{D_{N-1}(F; 0)} = \frac{N\gamma}{2} (x^2 - 1) + \frac{\gamma}{4} \log 2\pi + \frac{\gamma^2}{8} \log N - \log G(1 + \frac{\gamma}{2}) + o(1).$$

Finally, recall that we already argued that by rotation invariance of the law of the eigenvalues, $\mathbb{E}|\det(G_N - z)|^\gamma = \mathbb{E}|\det(G_N - x)|^\gamma$ for $|z| = x$, so by Lemma 2.1 (applied to the function $F(z) = 1$ which corresponds to $\gamma = 0$) we conclude that

$$\begin{aligned} \mathbb{E}|\det(G_N - z)|^\gamma &= \frac{N!}{Z_N} D_{N-1}(F; 0) \frac{D_{N-1}(F; \gamma)}{D_{N-1}(F; 0)} \\ &= N^{\frac{\gamma^2}{8}} e^{\frac{N\gamma}{2}(|z|^2 - 1)} \frac{(2\pi)^{\frac{\gamma}{4}}}{G(1 + \frac{\gamma}{2})} (1 + o(1)), \end{aligned}$$

which is the claim. \square

APPENDIX A. ORTHOGONAL POLYNOMIALS – PROOFS FOR THE RESULTS IN SECTION 2

In this appendix we prove Lemma 2.4 and Lemma 2.5. We begin with our proof of Lemma 2.4, which is essentially that of [3, the proof of Lemma 3.1].

Proof of Lemma 2.4. For $w \in \mathbb{C} \setminus (-\infty, x)$, let

$$h(w) = (w - x)^{\frac{\gamma}{2}} \int_x^{\overline{w}} (s - x)^{\frac{\gamma}{2}+k} e^{-Nws} ds,$$

where the roots are according to the principal branch, and the integration contour does not intersect $(-\infty, x)$. One has

$$\frac{\partial}{\partial \overline{w}} h(w) = |w - x|^\gamma (\overline{w} - x)^k e^{-N|w|^2},$$

so we see by (2.2) and Green's theorem (one can check that the first partial derivatives of h are continuous – the jumps in the roots along $(-\infty, x)$ cancel and derivatives at $w = x$ are zero for $\gamma > 0$) that for $k \leq j$

$$\begin{aligned} \frac{1}{\chi_j} \delta_{j,k} &= \int_{\mathbb{C}} p_j(w) \overline{w}^k |w - x|^\gamma e^{-N|w|^2} d^2 w \\ &= \int_{\mathbb{C}} p_j(w) (\overline{w} - x)^k |w - x|^\gamma e^{-N|w|^2} d^2 w \\ &= \lim_{r \rightarrow \infty} \int_{|w| \leq r} \frac{\partial}{\partial \overline{w}} [p_j(w) h(w)] d^2 w \\ &= \lim_{r \rightarrow \infty} \frac{1}{2i} \oint_{|w|=r} p_j(w) h(w) dw. \end{aligned}$$

We now wish to deform the $\{|w| = r\}$ contour into Σ . To do this, we note that for $|w| = r$

$$h(w) = (w - x)^{\frac{\gamma}{2}} \left(\int_x^{\overline{w} \times \infty} (s - x)^{\frac{\gamma}{2}+k} e^{-Nws} ds - \int_{\overline{w}}^{\overline{w} \times \infty} (s - x)^{\frac{\gamma}{2}+k} e^{-Nws} ds \right),$$

where again we take the contours to not intersect $(-\infty, x)$. The second integral is easily seen to be $\mathcal{O}(e^{-\frac{1}{2}|r|^2 N})$ uniformly on $\{|w| = r\}$. For the first integral we note that

$$\begin{aligned} (w - x)^{\frac{\gamma}{2}} \int_x^{\overline{w} \times \infty} (s - x)^{\frac{\gamma}{2}+k} e^{-Nws} ds &= N^{-\frac{\gamma}{2}-k-1} w^{-k-1} w^{-\frac{\gamma}{2}} (w - x)^{\frac{\gamma}{2}} e^{-Nw\overline{w}} \Gamma\left(\frac{\gamma}{2} + k + 1\right) \\ &= \frac{\pi \Gamma\left(\frac{\gamma}{2} + k + 1\right)}{N^{\frac{\gamma}{2}+k+1}} w^{-k} \frac{f(w)}{\pi w} \end{aligned}$$

which is an analytic function of w in $\mathbb{C} \setminus [0, x]$. We thus see by contour deformation and our bound on the second integral that if Σ is a simple closed contour encircling $[0, x]$ (not passing through any point of the interval)

$$\frac{1}{\chi_j} \delta_{j,k} = \frac{\pi \Gamma\left(\frac{\gamma}{2} + k + 1\right)}{N^{\frac{\gamma}{2}+k+1}} \oint_{\Sigma} p_j(w) w^{-k} f(w) \frac{dw}{2\pi i w},$$

which was precisely the claim. The only remaining issue is to consider the case where Σ passes through x . Let $\epsilon > 0$ and let Σ_ϵ be an indentation of Σ at x such that Σ_ϵ does not pass through x . We then have

$$\oint_{\Sigma} p_j(w) w^{-k} f(w) \frac{dw}{2\pi i w} = \oint_{\Sigma_\epsilon} p_j(w) w^{-k} f(w) \frac{dw}{2\pi i w} + \oint_{C_\epsilon} p_j(w) w^{-k} f(w) \frac{dw}{2\pi i w},$$

where $C_\epsilon = (\Sigma \setminus \Sigma_\epsilon) \cup (\Sigma_\epsilon \cup \Sigma)$. The first integral here is precisely what we want the left hand side to be for each $\epsilon > 0$ and since our integrand is bounded at x (it actually vanishes at x) we see that as $\epsilon \rightarrow 0$, the second integral vanishes. This concludes the proof. \square

We now turn to the proof of Lemma 2.5.

Proof of Lemma 2.5. The statement about $(q_j)_{j=0}^\infty$ being well defined is that $\widehat{D}_l \neq 0$ for all l . To see this, note that we can write (using elementary column operations on the determinant)

$$\widehat{D}_j = \prod_{k=0}^j \frac{1}{\chi_k} \begin{vmatrix} \oint_{\Sigma} p_0(w) f(w) \frac{dw}{2\pi i 2} & \cdots & \oint_{\Sigma} p_j(w) f(w) \frac{dw}{2\pi i w} \\ \vdots & \ddots & \vdots \\ \oint_{\Sigma} p_0(w) w^{-j} f(w) \frac{dw}{2\pi i w} & \cdots & \oint_{\Sigma} p_j(w) w^{-j} f(w) \frac{dw}{2\pi i w} \end{vmatrix}.$$

By Lemma 2.2, this determinant is upper triangular with diagonal elements $\frac{1}{\pi} \frac{N^{1+\frac{\gamma}{2}+k}}{\Gamma(1+\frac{\gamma}{2}+k)} \frac{1}{\chi_k}$. We conclude that

$$(A.1) \quad \widehat{D}_j = \prod_{k=0}^j \left(\chi_k^{-2} \frac{1}{\pi} \frac{N^{1+\frac{\gamma}{2}+k}}{\Gamma(1+\frac{\gamma}{2}+k)} \right) > 0$$

which implies that q_j is well defined for each j .

Let us next consider $\widehat{\chi}_j$. From (2.7) (the definition of q_j) and (A.1) we see that

$$\widehat{\chi}_j = \sqrt{\frac{\pi \Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2} + j + 1}}} \sqrt{\frac{\widehat{D}_{j-1}}{\widehat{D}_j}} = \frac{\pi \Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2} + j + 1}} \chi_j$$

which is precisely (2.10).

Let us consider finally the orthogonality relation (2.9). We note that for $0 \leq k \leq j$, we find from Lemma 2.4 and (2.10) that

$$\begin{aligned} \oint_{\Sigma} p_j(w) q_k(w^{-1}) f(w) \frac{dw}{2\pi i w} &= \widehat{\chi}_k \oint_{\Sigma} p_j(w) w^{-k} f(w) \frac{dw}{2\pi i w} \\ &= \widehat{\chi}_k \delta_{j,k} \frac{1}{\pi} \frac{N^{1+\frac{\gamma}{2}+k}}{\Gamma(1+\frac{\gamma}{2}+k)} \frac{1}{\chi_k} \\ &= \delta_{j,k}. \end{aligned}$$

For $k > j \geq 0$, we see again from the definition of q_j that

$$\begin{aligned} \oint_{\Sigma} w^j q_k(w^{-1}) f(w) \frac{dw}{2\pi i w} &= \sqrt{\frac{\pi \Gamma(\frac{\gamma}{2} + k + 1)}{N^{\frac{\gamma}{2} + k + 1}}} \frac{1}{\sqrt{\widehat{D}_{k-1} \widehat{D}_k}} \\ &\quad \times \begin{vmatrix} \oint_{\Sigma} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^{k-1} f(s) \frac{ds}{2\pi i s} & \oint_{\Sigma} w^j f(w) \frac{dw}{2\pi i w} \\ \vdots & & \vdots & \vdots \\ \oint_{\Sigma} s^{-k} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^{-1} f(s) \frac{ds}{2\pi i s} & \oint_{\Sigma} w^{j-k} f(w) \frac{dw}{2\pi i w} \end{vmatrix} \\ &= 0, \end{aligned}$$

since the determinant has two identical columns. This concludes the proof of the lemma. \square

APPENDIX B. PROOF OF THE DIFFERENTIAL IDENTITY

In this appendix we prove our differential identity – Lemma 3.3. To prove it, we need to recall suitable recursion relations for the polynomials as well as the Christoffel-Darboux identity for the polynomials p and q . While this is a standard fact and the proof we present below is a trivial modification of that in [10, Lemma 2.3], there are some cosmetic differences due to the fact that $\chi_j \neq \hat{\chi}_j$, so we choose to present a proof here. We start with some recurrence relations for the polynomials – this is very similar to [10, Lemma 2.2].

Lemma B.1. *The following identities hold: for any $n \in \mathbb{Z}_+$*

$$(B.1) \quad \hat{\chi}_n w p_n(w) = \hat{\chi}_{n+1} p_{n+1}(w) - p_{n+1}(0) w^{n+1} q_{n+1}(w^{-1}),$$

$$(B.2) \quad \chi_n w^{-1} q_n(w^{-1}) = \chi_{n+1} q_{n+1}(w^{-1}) - q_{n+1}(0) w^{-n-1} p_{n+1}(w),$$

$$(B.3) \quad \hat{\chi}_{n+1} w^{-1} q_n(w^{-1}) = \hat{\chi}_n q_{n+1}(w^{-1}) - q_{n+1}(0) \frac{\hat{\chi}_n}{\chi_n} w^{-n} p_n(w),$$

$$(B.4) \quad \chi_n \hat{\chi}_n = \chi_{n+1} \hat{\chi}_{n+1} - p_{n+1}(0) q_{n+1}(0).$$

Proof. Let

$$g(w) := p_n(w) - a w^{-1} p_{n+1}(w) - b w^n q_{n+1}(w^{-1}).$$

We want to choose a and b so that g vanishes. We first show that with a good choice of b , g is actually a polynomial in w so that we can express it in terms of the polynomials p_k (with $k \leq n$). We then show that by choosing a the correct way, the coefficients of p_k vanish for all $k \leq n$.

We thus begin by making sure that the term of order w^{-1} vanishes (there are no lower order terms in g). For this, we note that the coefficient of w^{-1} in $g(w)$ is $-a p_{n+1}(0) - b \hat{\chi}_{n+1}$, so we choose $b = -\frac{a p_{n+1}(0)}{\hat{\chi}_{n+1}}$. Thus g is a polynomial in w and its degree is at most n . To expand it in the basis (p_k) , we know from (2.9) that is enough to evaluate $\oint_{\Sigma} g(w) q_l(w^{-1}) f(w) \frac{dw}{2\pi i w}$ for $l \leq n$. We have from (2.9)

- $\oint_{\Sigma} p_n(w) q_l(w^{-1}) f(w) \frac{dw}{2\pi i w} = \delta_{l,n}.$
- $\oint_{\Sigma} w^{-1} p_{n+1}(w) q_l(w^{-1}) f(w) \frac{dw}{2\pi i w} = \delta_{l,n} \frac{\hat{\chi}_n}{\hat{\chi}_{n+1}}.$
- $\oint_{\Sigma} w^n q_{n+1}(w^{-1}) q_l(w^{-1}) f(w) \frac{dw}{2\pi i w} = 0.$

Therefore if we choose $a = \frac{\hat{\chi}_{n+1}}{\chi_n}$, we see that for all $l \leq n$, $\oint_{\Sigma} g(w) q_l(w^{-1}) f(w) \frac{dw}{2\pi i w} = 0$ implying that $g(w) = 0$ for all w . This gives (B.1). The proof of (B.2) is similar, and one can obtain (B.3) by combining the first two recurrence relations. To obtain (B.4) one inspects the coefficient of w^{n+1} in (B.1). \square

This lets us prove the Christoffel-Darboux identity.

Lemma B.2 (Christoffel-Darboux). *For any $w, u \neq 0, n \in \mathbb{N}$, we have*

$$(B.5) \quad (1 - u^{-1}w) \sum_{k=0}^{n-1} p_k(w) q_k(u^{-1}) = u^{-n} p_n(u) w^n q_n(w^{-1}) - p_n(w) q_n(u^{-1}).$$

In particular, for any $w \neq 0$ and $n \in \mathbb{N}$,

$$(B.6) \quad \sum_{k=0}^{n-1} p_k(w) q_k(w^{-1}) = -n p_n(w) q_n(w^{-1}) + w (q_n(w^{-1}) \partial_w p_n(w) - p_n(w) \partial_w q_n(w^{-1})).$$

Proof. Using (B.1) and (B.3), we have

$$\begin{aligned} u^{-1}wp_k(w)q_k(u^{-1}) &= (wp_k(w))(u^{-1}q_k(u^{-1})) \\ &= \left[\frac{\widehat{\chi}_{k+1}}{\widehat{\chi}_k} p_{k+1}(w) - \frac{p_{k+1}(0)}{\widehat{\chi}_k} w^{k+1} q_{k+1}(w^{-1}) \right] \left[\frac{\widehat{\chi}_k}{\widehat{\chi}_{k+1}} q_{k+1}(u^{-1}) - \frac{q_{k+1}(0)}{\widehat{\chi}_{k+1}} \frac{\widehat{\chi}_k}{\chi_k} u^{-k} p_k(u) \right], \end{aligned}$$

and hence

$$\begin{aligned} (1 - u^{-1}w)p_k(w)q_k(u^{-1}) &= p_k(w)q_k(u^{-1}) - p_{k+1}(w)q_{k+1}(u^{-1}) \\ &\quad + \left(\frac{w}{u}\right)^{k+1} \left[\frac{q_{k+1}(w^{-1})}{\widehat{\chi}_{k+1}} p_{k+1}(0) u^{k+1} q_{k+1}(u^{-1}) + \frac{up_k(u)}{\chi_k} q_{k+1}(0) w^{-k-1} p_k(w) \right. \\ &\quad \left. - \frac{p_{k+1}(0)q_{k+1}(0)}{\chi_k \widehat{\chi}_{k+1}} up_k(u) q_{k+1}(w^{-1}) \right]. \end{aligned}$$

But from (B.1), (B.2) and (B.4), we see that

$$\begin{aligned} \frac{q_{k+1}(w^{-1})}{\widehat{\chi}_{k+1}} p_{k+1}(0) u^{k+1} q_{k+1}(u^{-1}) &= q_{k+1}(w^{-1}) \left[p_{k+1}(u) - \frac{\widehat{\chi}_k}{\widehat{\chi}_{k+1}} up_k(u) \right] \\ \frac{up_k(u)}{\chi_k} q_{k+1}(0) w^{-k-1} p_k(w) &= up_k(u) \left[\frac{\chi_{k+1}}{\chi_k} q_{k+1}(w^{-1}) - w^{-1} q_k(w^{-1}) \right] \\ - \frac{p_{k+1}(0)q_{k+1}(0)}{\chi_k \widehat{\chi}_{k+1}} up_k(u) q_{k+1}(w^{-1}) &= \left(\frac{\widehat{\chi}_k}{\widehat{\chi}_{k+1}} - \frac{\chi_{k+1}}{\chi_k} \right) up_k(u) q_{k+1}(w^{-1}) \end{aligned}$$

and therefore

$$\begin{aligned} (1 - u^{-1}w)p_k(w)q_k(u^{-1}) &= p_k(w)q_k(u^{-1}) - p_{k+1}(w)q_{k+1}(u^{-1}) \\ &\quad + \left(\frac{w}{u}\right)^{k+1} p_{k+1}(u)q_{k+1}(w^{-1}) - \left(\frac{w}{u}\right)^k p_k(u)q_k(w^{-1}). \end{aligned}$$

(B.5) now follows by taking the sum from $k = 0$ to $k = n - 1$. (B.6) follows from dividing (B.5) by $(1 - u^{-1}w)$ and letting $u \rightarrow w$. \square

We can finally turn to our differential identity. This is very similar to corresponding proofs in [10, 12, 18].

Proof of Lemma 3.3. We begin by noting that from Lemma 2.2

$$(B.7) \quad \partial_\gamma \log D_{N-1}(F; \gamma) = -2 \sum_{j=0}^{N-1} \frac{\partial_\gamma \chi_j}{\chi_j},$$

where the smoothness of χ_j and D_{N-1} as functions of γ follows e.g. from the determinantal representation (2.4). It follows from (2.9) that

$$\oint_{\Sigma} [\partial_\gamma p_j(w)] q_j(w^{-1}) f(w) \frac{dw}{2\pi i w} = \frac{\partial_\gamma \chi_j}{\chi_j} \quad \text{and} \quad \oint_{\Sigma} p_j(w) [\partial_\gamma q_j(w^{-1})] f(w) \frac{dw}{2\pi i w} = \frac{\partial_\gamma \widehat{\chi}_j}{\widehat{\chi}_j}.$$

Moreover, we see from (2.10) that

$$(B.8) \quad \frac{\partial_\gamma \widehat{\chi}_j}{\widehat{\chi}_j} = \frac{\partial_\gamma \chi_j}{\chi_j} + \partial_\gamma \log \Gamma \left(\frac{\gamma}{2} + j + 1 \right) - \frac{1}{2} \log N.$$

We can thus rewrite (B.7) as

$$(B.9) \quad \begin{aligned} \partial_\gamma \log D_{N-1}(F; \gamma) = & - \sum_{j=0}^{N-1} \oint_{\Sigma} \partial_\gamma (p_j(w) q_{j-1}(w^{-1})) f(w) \frac{dw}{2\pi i w} \\ & + \partial_\gamma \sum_{j=0}^{N-1} \log \frac{\Gamma(\frac{\gamma}{2} + j + 1)}{N^{\frac{\gamma}{2}}}. \end{aligned}$$

Applying the Christoffel-Darboux identity (B.6) and the orthogonality relations (2.9), we have

$$(B.10) \quad \begin{aligned} & \oint_{\Sigma} \left[\partial_\gamma \sum_{n=0}^{N-1} p_n(w) q_n(w^{-1}) \right] f(w) \frac{dw}{2\pi i w} \\ &= \oint_{\Sigma} \partial_\gamma [-N p_N(w) q_N(w^{-1}) + w(q_N(w^{-1}) \partial_w p_N(w) - p_N(w) \partial_w q_N(w^{-1}))] f(w) \frac{dw}{2\pi i w} \\ &= -N \left[\frac{\partial_\gamma \chi_N}{\chi_N} + \frac{\partial_\gamma \hat{\chi}_N}{\hat{\chi}_N} \right] + \oint_{\Sigma} [\partial_\gamma (q_N(w^{-1}) \partial_w p_N(w) - p_N(w) \partial_w q_N(w^{-1}))] w f(w) \frac{dw}{2\pi i w} \\ &= \underbrace{\oint_{\Sigma} \frac{\partial q_N(w^{-1})}{\partial \gamma} \frac{\partial p_N(w)}{\partial w} f(w) \frac{dw}{2\pi i}}_{=: I_1} - \underbrace{\oint_{\Sigma} \frac{\partial q_N(w^{-1})}{\partial w} \frac{\partial p_N(w)}{\partial \gamma} f(w) \frac{dw}{2\pi i}}_{=: I_2}. \end{aligned}$$

Note that

$$\partial_w f(w) = f(w) \left[\frac{\gamma}{2} \frac{1}{w-x} - \frac{\gamma}{2} \frac{1}{w} - Nx \right],$$

and therefore by integration by parts and orthogonality we get

$$(B.11) \quad \begin{aligned} I_1 &= \oint_{\Sigma} \frac{\partial q_N(w^{-1})}{\partial \gamma} \frac{\partial p_N(w)}{\partial w} f(w) \frac{dw}{2\pi i} \\ &= - \oint_{\Sigma} p_N(w) w \partial_w \left[\frac{\partial q_N(w^{-1})}{\partial \gamma} \right] f(w) \frac{dw}{2\pi i w} \\ &\quad - \oint_{\Sigma} p_N(w) \frac{\partial q_N(w^{-1})}{\partial \gamma} f(w) \left[\frac{\gamma}{2} \frac{x}{w-x} - Nxw \right] \frac{dw}{2\pi i w}. \end{aligned}$$

To analyze this further, we begin by noting that by orthogonality

$$(B.12) \quad - \oint_{\Sigma} p_N(w) w \partial_w \left[\frac{\partial q_N(w^{-1})}{\partial \gamma} \right] f(w) \frac{dw}{2\pi i w} = N \frac{\partial_\gamma \hat{\chi}_N}{\hat{\chi}_N}.$$

Next we point out that

$$\begin{aligned} & - \frac{\gamma}{2} \oint_{\Sigma} p_N(w) \frac{\partial q_N(w^{-1})}{\partial \gamma} \frac{x}{w-x} f(w) \frac{dw}{2\pi i w} \\ &= - \frac{\gamma}{2} \oint_{\Sigma} p_N(w) w^{-N+1} \left[\frac{\partial}{\partial \gamma} w^{N-1} q_N(w^{-1}) \right] \frac{x f(w)}{w-x} \frac{dw}{2\pi i w} \\ &= - \frac{\gamma}{2} x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} \oint_{\Sigma} p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} \\ &\quad - \frac{\gamma}{2} \oint_{\Sigma} p_N(w) w^{-N+1} \frac{\partial_\gamma (w^{N-1} q_N(w^{-1}) - x^{N-1} q_N(x^{-1}))}{w-x} x f(w) \frac{dw}{2\pi i w}. \end{aligned}$$

One can easily check that

$$w^{-N+1} \frac{\partial_\gamma(w^{N-1}q_N(w^{-1}) - x^{N-1}q_N(x^{-1}))}{w-x}x = -\partial_\gamma \widehat{\chi}_N w^{-N} + \mathcal{P}_{N-1}(w^{-1}),$$

where \mathcal{P}_{N-1} is a polynomial of degree at most $N-1$. Thus by orthogonality,

$$(B.13) \quad \begin{aligned} & -\frac{\gamma}{2} \oint_{\Sigma} p_N(w) \frac{\partial q_N(w^{-1})}{\partial \gamma} \frac{x}{w-x} f(w) \frac{dw}{2\pi i w} \\ & = -\frac{\gamma}{2} x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} \oint_{\Sigma} p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} + \frac{\gamma}{2} \frac{\partial_\gamma \widehat{\chi}_N}{\widehat{\chi}_N}. \end{aligned}$$

To analyze I_1 , we still have one term left. We recall from (B.1) that

$$w p_N(w) = \frac{\widehat{\chi}_{N+1}}{\widehat{\chi}_N} p_{N+1}(w) - \frac{p_{N+1}(0)}{\widehat{\chi}_N} w^{N+1} q_{N+1}(w^{-1})$$

so by orthogonality

$$(B.14) \quad \oint_{\Sigma} w p_N(w) \frac{\partial q_N(w^{-1})}{\partial \gamma} f(w) \frac{dw}{2\pi i w} = -\frac{p_{N+1}(0)}{\widehat{\chi}_N} \frac{\partial_\gamma q_N(0)}{\chi_{N+1}}.$$

Plugging (B.12), (B.13), and (B.14) into (B.11), we see that

$$(B.15) \quad I_1 = \left(N + \frac{\gamma}{2}\right) \frac{\partial_\gamma \widehat{\chi}_N}{\chi_N} - \frac{\gamma}{2} x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} \oint_{\Sigma} p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} - N x \frac{p_{N+1}(0)}{\chi_{N+1}} \frac{\partial_\gamma q_N(0)}{\widehat{\chi}_N}.$$

Recalling that we write $p_j(w) = \chi_j w^j + \kappa_j w^{j-1} + \mathcal{O}(w^{j-2})$, we note for I_2 that we have

$$w \partial_\gamma p_N(w) = \frac{\partial_\gamma \chi_N}{\chi_{N+1}} p_{N+1}(w) + \left(\frac{\partial_\gamma \kappa_N}{\chi_N} - \frac{\partial_\gamma \chi_N}{\chi_{N+1}} \frac{\kappa_{N+1}}{\chi_N} \right) p_N(w) + \mathcal{O}(w^{N-1}).$$

With this identity, arguments similar to those we used to analyze I_1 lead to

$$(B.16) \quad \begin{aligned} I_2 &= -N \frac{\partial_\gamma \chi_N}{\chi_N} - \frac{\gamma}{2} \frac{\partial p_N(x)}{\partial \gamma} x \oint_{\Sigma} \frac{q_N(w^{-1}) f(w)}{w-x} \frac{dw}{2\pi i w} + N x \oint_{\Sigma} q_N(w^{-1}) w \left[\frac{\partial}{\partial \gamma} p_N(w) \right] f(w) \frac{dw}{2\pi i w} \\ &= -N \frac{\partial_\gamma \chi_N}{\chi_N} - \frac{\gamma}{2} \frac{\partial p_N(x)}{\partial \gamma} x \oint_{\Sigma} \frac{q_N(w^{-1}) f(w)}{w-x} \frac{dw}{2\pi i w} + N x \left(\frac{\partial_\gamma \kappa_N}{\chi_N} - \frac{\partial_\gamma \chi_N}{\chi_N} \frac{\kappa_{N+1}}{\chi_{N+1}} \right). \end{aligned}$$

Combining (B.15) and (B.16) with (B.10) and (B.9), we find

$$\begin{aligned} & \partial_\gamma \log D_{N-1}(F; \gamma) \\ &= -\left(N + \frac{\gamma}{2}\right) \frac{\partial_\gamma \widehat{\chi}_N}{\widehat{\chi}_N} + \frac{\gamma}{2} x^N \frac{\partial q_N(x^{-1})}{\partial \gamma} \oint_{\Sigma} p_N(w) w^{-N+1} \frac{f(w)}{w-x} \frac{dw}{2\pi i w} - N x \frac{p_{N+1}(0)}{\chi_{N+1}} \frac{\partial_\gamma q_N(0)}{\widehat{\chi}_N} \\ & \quad - N \frac{\partial_\gamma \chi_N}{\chi_N} - \frac{\gamma}{2} \frac{\partial p_N(x)}{\partial \gamma} x \oint_{\Sigma} \frac{q_N(w^{-1}) f(w)}{w-x} \frac{dw}{2\pi i w} + N x \left(\frac{\partial_\gamma \kappa_N}{\chi_N} - \frac{\partial_\gamma \chi_N}{\chi_N} \frac{\kappa_{N+1}}{\chi_{N+1}} \right) \\ & \quad + \partial_\gamma \sum_{j=0}^{N-1} \log \frac{\Gamma\left(\frac{\gamma}{2} + j + 1\right)}{N^{\frac{\gamma}{2}}}. \end{aligned}$$

which is the claim. □

APPENDIX C. ASYMPTOTIC ANALYSIS OF THE RIEMANN-HILBERT PROBLEM – PROOFS FOR SECTION 4

In this appendix, we give proofs related to the asymptotic analysis of our Riemann-Hilbert problem. We begin with Lemma 4.1.

Proof of Lemma 4.1. The fact that Σ is a smooth, simple closed loop, encircling $[0, x]$ and passing only through x follows e.g. from writing

$$\Sigma = \left\{ \left(u, \sqrt{x^2 e^{2x(u-x)} - u^2} \right) : u_0 \leq u \leq x \right\} \cup \left\{ \left(u, -\sqrt{x^2 e^{2x(u-x)} - u^2} \right) : u_0 \leq u \leq x \right\},$$

where u_0 is the unique negative solution to the equation $x(u-x) + \log x - \log |u|$ (one can easily check that this equation has only one negative solution and for $u \in (0, x]$, the only solution is $u = x$). The fact that Σ is inside the unit circle is obvious from (4.3) – the definition of Σ .

The fact that $\operatorname{Re}(xw + \ell - \log w)$ is positive in $\operatorname{Int}(\Sigma)$ follows from the definition of Σ and evaluating $\operatorname{Re}(xw + \ell - \log w)$ at $w = 0$ (recall that Σ encircles $[0, x]$). To see that $\operatorname{Re}(xw + \ell - \log w)$ is negative in $\operatorname{Ext}(\Sigma) \cap \{|w| \leq 1\}$, note first that on the unit circle, $\operatorname{Re}(xw + \ell - \log w) = x \operatorname{Re}(w) + \ell \leq x + \log x - x^2 < 0$ for $x < 1$ (this also proves the claim of the uniform negative bound on the unit circle). Then we note that as $\operatorname{Re}(xw + \ell - \log w)$ is zero on Σ and its only critical point is $w = 1/x > 1$, there can't be any points in $\operatorname{Ext}(\Sigma) \cap \{|w| \leq 1\}$ where it's positive (one of them would have to be a critical point). \square

Let us then move on to the Riemann-Hilbert problem that S satisfies.

Proof of Lemma 4.2. Analyticity and continuity of boundary values is clear from the corresponding properties for Y and the definition of S . The jump condition across $\{|w| = 1\}$ is also immediate from the definitions. Consider then the jump across Σ . From the definition of S and T , we have for $w \in \Sigma \setminus \{x\}$

$$\begin{aligned} S_+(w) &= S_-(w) \begin{pmatrix} 1 & 0 \\ -w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} e^{-(N+k)(xw+\ell-\log w)} & (w-x)^{\frac{\gamma}{2}} w^{-\frac{\gamma}{2}} e^{kxw} \\ 0 & e^{(N+k)(xw+\ell-\log w)} \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 \\ -w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{-kxw} e^{-(N+k)(xw+\ell-\log w)} & 1 \end{pmatrix} \\ &= S_-(w) \begin{pmatrix} 0 & (w-x)^{\frac{\gamma}{2}} w^{-\frac{\gamma}{2}} e^{kxw} \\ -(w-x)^{-\frac{\gamma}{2}} w^{\frac{\gamma}{2}} e^{-kxw} & 0 \end{pmatrix}. \end{aligned}$$

For the jump across $(0, x)$, note that the only term contributing to the branch cut is $(w-x)^{-\frac{\gamma}{2}}$. The claimed jump is easily obtained by looking at the jump of this function.

For the behavior near zero, we note that from the definition of S , we have $\lim_{w \rightarrow 0} S(w) = T(0)$. The behavior near x comes from the fact that T is bounded near x . The normalization at infinity follows from the corresponding one for T . \square

Our next task is to describe the RHP that the local parametrix $P^{(x)}$ satisfies.

Proof of Lemma 4.4. One can easily check from the definition of $\zeta(w)$ that

$$w \mapsto N^{\frac{\gamma}{2}-1} \frac{x e^{kx^2} (1-x^2)^{\frac{\gamma}{2}-1}}{\Gamma(\frac{\gamma}{2})(w-x)} - \left(\frac{w\zeta(w)}{w-x} \right)^{\frac{\gamma}{2}} \frac{e^{kxw}}{\zeta(w)\Gamma(\frac{\gamma}{2})}.$$

is analytic (in a small enough but fixed neighborhood of x). Thus the possible singularities of $P^{(x)}$ come from those of $P^{(\infty)}$, $(w-x)^{-\frac{\gamma}{2}}$, and the branch cut of $\Gamma(\frac{\gamma}{2}, \zeta(w))$. We conclude that $P^{(x)}$ indeed satisfies the claimed analyticity condition.

For the jump conditions, we note that on $\Sigma \setminus \{x\}$, the only singularity comes from $P^{(\infty)}$. Thus the jump condition on $\Sigma \setminus \{x\}$ comes immediately from the definition of $P^{(x)}$ and the jump condition of $P^{(\infty)}$. For the jump on $(0, x)$, we note the $P^{(\infty)}$ has no singularity here so everything comes from the upper triangular matrix. Here the analytic term does not contribute and we simply need the following calculation (which is easy to check from the representation of $\Gamma(\nu, \zeta)$ in terms of $\gamma^*(\nu, \zeta)$):

$$\begin{aligned} & \left(w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{kxw} \frac{\Gamma(\frac{\gamma}{2}, \zeta(w)) e^{\zeta(w)}}{\Gamma(\frac{\gamma}{2})} \right)_+ - \left(w^{\frac{\gamma}{2}}(w-x)^{-\frac{\gamma}{2}} e^{kxw} \frac{\Gamma(\frac{\gamma}{2}, \zeta(w)) e^{\zeta(w)}}{\Gamma(\frac{\gamma}{2})} \right)_- \\ &= w^{\frac{\gamma}{2}} e^{\zeta(w)} e^{kxw} \left[(w-x)_+^{-\frac{\gamma}{2}} - (w-x)_-^{-\frac{\gamma}{2}} \right] \\ &= -2i \sin \frac{\pi\gamma}{2} |w|^{\frac{\gamma}{2}} |w-x|^{-\frac{\gamma}{2}} e^{kxw} e^{\zeta(w)} \end{aligned}$$

from which we find that for $w \in (0, x) \cap U$

$$\begin{aligned} \left[P_-^{(x)}(w) \right]^{-1} P_+^{(x)}(w) &= \begin{pmatrix} 0 & -e^{kxw} \\ e^{-kxw} & 0 \end{pmatrix} \begin{pmatrix} 1 & -2i \sin \frac{\pi\gamma}{2} |w|^{\frac{\gamma}{2}} |w-x|^{-\frac{\gamma}{2}} e^{kxw} e^{\zeta(w)} \\ 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & e^{kxw} \\ -e^{-kxw} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 2i \sin \frac{\pi\gamma}{2} |w|^{\frac{\gamma}{2}} |w-x|^{-\frac{\gamma}{2}} e^{-kxw} e^{\zeta(w)} & 1 \end{pmatrix}, \end{aligned}$$

which is (4.15).

Let us consider now the asymptotic behavior at x . We consider the behavior first as $w \rightarrow x$ from $\text{Ext}(\Sigma)$. We have

$$\begin{aligned} P^{(x)}(w) &= \begin{pmatrix} 1 & \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & 0 \\ 0 & \mathcal{O}(|w-x|^{\frac{\gamma}{2}}) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & \mathcal{O}(1) \\ 0 & \mathcal{O}(|w-x|^{\frac{\gamma}{2}}) \end{pmatrix}. \end{aligned}$$

Similarly as $w \rightarrow x$ from $\text{Int}(\Sigma) \setminus [0, x]$ we have

$$\begin{aligned} P^{(x)}(w) &= \begin{pmatrix} 1 & \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & e^{kxw} \\ -e^{-kxw} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(|w-x|^{-\frac{\gamma}{2}}) & e^{kxw} \\ -e^{-kxw} & 0 \end{pmatrix} \end{aligned}$$

which was the claim.

For the matching condition, we'll make use of the following expansion for $\Gamma(\nu, \zeta)$ (see e.g. [24, Section 4.2]): for any $p \in \mathbb{Z}_+$,

$$(C.1) \quad \frac{\Gamma(\frac{\gamma}{2}, \zeta)}{\Gamma(\frac{\gamma}{2})} e^{\zeta} = \zeta^{\frac{\gamma}{2}-1} \left(\sum_{k=0}^p \frac{1}{\Gamma(\frac{\gamma}{2}-k)} \zeta^{-k} + \mathcal{O}(\zeta^{-p-1}) \right)$$

as $\zeta \rightarrow \infty$ for $|\arg \zeta| < \frac{3\pi}{2} - \delta$ for any fixed $\delta > 0$. We can make use of this since $|\zeta| \asymp N$ uniformly on ∂U (and also uniformly in γ) so we find from (C.1) with $p = 0$ and Remark 4.3, that uniformly for $w \in \partial U$ and uniformly in $\gamma \in (0, 2)$

$$P^{(x)}(w) \left[\widehat{P}^{(\infty)}(w) \right]^{-1} = \begin{pmatrix} 1 & \mathcal{O}(|\zeta(w)|^{\frac{\gamma}{2}-2}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{O}(N^{\frac{\gamma}{2}-2}) \\ 0 & 1 \end{pmatrix}$$

which yields the claim. \square

We now turn to proving that R is a solution to the RHP we claimed.

Proof of Lemma 4.5. The proof is largely standard. We note that by construction, the branch cuts of the parametrices cancel with those of S , and the only jumps are across ∂U and the unit circle. For analyticity, one still needs to check that there is no isolated singularity at x . Using (4.8) and (4.16), one sees that the singularity of R at x is at most of strength $\mathcal{O}(|w-x|^{-\gamma/2})$ so for $\gamma < 2$ it can't be a pole (or essential) and it must be removable.

Thus continuity of the boundary values and the structure of the jump matrices follow directly from the relevant definitions and from (4.5). The normalization at infinity also follows from the asymptotic behavior of S and $\widehat{P}^{(\infty)}$ at infinity. The estimates for the jump matrices follow from (4.17) and Lemma 4.1. \square

We conclude with the proof of the asymptotic behavior of R .

Proof of Lemma 4.6. Again, most of the proof is standard and surely obvious for experts, but for the convenience of the reader, we offer a sketch of a proof here. For our proof (which follows reasoning from [18, 10, 12]) we find it convenient to generalize slightly, namely we consider the RHP from Lemma 4.5 for complex γ for which $\operatorname{Re}(\gamma) \in (-2, 2)$. The local and global parametrices are well defined for complex γ as well and an identical argument shows that $\sup_{w \in U} |\Delta_R(w)| = \mathcal{O}(N^{-2+\frac{1}{2}\operatorname{Re}(\gamma)})$ (while the proof of the asymptotic expansion of the incomplete gamma function is given in [24, Section 4.2] only for real γ , the proof works with obvious modifications also for complex γ) and $\sup_{|w|=1} |\Delta_R(w)| = \mathcal{O}(e^{-cN})$ also in the complex case. Moreover, if γ is restricted to some compact subset of $\{s \in \mathbb{C} : \operatorname{Re}(s) \in (-2, 2)\}$, these estimates are uniform in γ . They are also uniform in x in a compact subset of $(0, 1)$.

We now recall how one sees that this RHP has a unique solution for complex γ as well. The uniqueness can be proved the standard way. To see existence, we introduce some (standard) notation: for $w \in \mathbb{C} \setminus \Gamma_R$, let

$$C(f) := \int_{\Gamma_R} \frac{f(s)}{s-w} \frac{ds}{2\pi i}$$

and let $C_{\Delta_R}(f) = C_-(f\Delta_R)$, where $C_-(f)(w) = \lim_{z \rightarrow w} C(f)(z)$ as z approaches $w \in \Gamma_R$ from the $-$ side of Γ_R . Since $C_- : L^2(\Gamma_R) \rightarrow L^2(\Gamma_R)$ is a bounded operator (see [13, Appendix A] and the references therein), our estimate on the jump matrix of R , namely $\|\Delta_R\|_{L^\infty(\Gamma_R)} = \mathcal{O}(N^{-2+\frac{1}{2}\operatorname{Re}(\gamma)})$ implies that the operator norm of C_{Δ_R} is $\mathcal{O}(N^{-2+\frac{\operatorname{Re}(\gamma)}{2}})$, and therefore for large enough N (large enough being independent of γ if γ is in the aforementioned compact set and independent of x if it is a fixed compact subset of $(0, 1)$), $I - C_{\Delta_R}$ is invertible. Arguing as in [13, the proof of Theorem 7.8] (though in a slightly inverted order since we don't know the existence of a solution) one can check that

$$(C.2) \quad R = I + C[\Delta_R + (I - C_{\Delta_R})^{-1}(C_{\Delta_R}(I))\Delta_R]$$

is a solution to the problem. Moreover, one can check that this implies that R can also be represented in terms of its boundary values:

$$(C.3) \quad R(w) = I + (C_{\Delta_R}R_-)(w) = I + \int_{\Gamma_R} \frac{R_-(s)\Delta_R(s)}{s-w} \frac{ds}{2\pi i}.$$

To get a hold of the asymptotic behavior of R , we note one consequence of the definition (C.2) is that $R_- - I = (1 - C_{\Delta_R})^{-1}C_{\Delta_R}(I)$. Since the norm of C_{Δ_R} is of order $N^{-2+\frac{1}{2}\text{Re}(\gamma)}$, we see from this that

$$(C.4) \quad \|R_- - I\|_{L^2(\Gamma_R)} \leq \|(I - C_{\Delta_R})^{-1}\|_{L^2(\Gamma_R) \rightarrow L^2(\Gamma_R)} \|C_{\Delta_R}(I)\|_{L^2(\Gamma_R)} = \mathcal{O}\left(N^{-2+\frac{1}{2}\text{Re}(\gamma)}\right).$$

Let us now fix $\delta > 0$ and let w be at distance at least δ from Γ_R . Then applying (C.4) to (C.3) and using Cauchy-Schwarz, we see that

$$\begin{aligned} |R(w) - I| &\leq |(C_{\Delta_R}I)(w)| + |(C_{\Delta_R}[R_- - I])(w)| \\ &= \mathcal{O}(\|C_{\Delta_R}\|_{L^\infty(\Gamma_R)}) + \mathcal{O}(\|R_- - I\|_{L^2(\Gamma_R)} \|\Delta_R\|_{L^2(\Gamma_R)}) \\ &= \mathcal{O}\left(N^{-2+\frac{1}{2}\text{Re}(\gamma)}\right), \end{aligned}$$

where the implied constants depend on δ , but are uniform in γ (when restricted to a compact set). This bound can be extended to points w close to Γ_R with the standard contour deformation argument – see [13, Corollary 7.9]. To conclude the proof of (4.21), note that we have from (C.3) that $\lim_{w \rightarrow \infty} w(R(w) - I) = -\int_{\Gamma_R} R_-(s)\Delta_R(s)\frac{ds}{2\pi i}$, for which repeating our previous argument shows the claim.

We now move onto the proof of (4.22) for which γ being complex will be of use – in particular, if we are able to show that $R(w)$ is an analytic function of γ , then Cauchy's integral formula combined with (4.21) will give (4.22). We note that going back in our chain of transformations, the existence of R lets us define the matrix Y in terms of R , the parametrices, and our transformations also for complex γ . Moreover, the RHP for R induces a RHP for Y as well and this RHP is precisely the one appearing in Lemma 3.2 with the difference that for general γ , the boundary values of Y are only continuous on $\Sigma \setminus \{x\}$. One can check that for $\text{Re}(\gamma) \in (-2, 2)$, the function Y obtained from R satisfies

$$(C.5) \quad Y(w) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|w - x|^{-1+\delta}) \\ \mathcal{O}(1) & \mathcal{O}(|w - x|^{-1+\delta}) \end{pmatrix},$$

for some fixed $\delta > 0$, and if one adds this condition to the RHP, then a standard argument shows that this problem has a unique solution.

It is then another standard argument (using the jump condition of Y , its asymptotic behavior, Liouville's theorem, and some regularity properties of the Cauchy transform – we omit the details) that even for complex γ , the polynomials $p_{N+k}(w)$ and $q_{N+k-1}(w^{-1})$ must exist and Y_{N+k} is given by (3.1) in terms of these polynomials. More precisely, for complex γ , one has

$$\frac{1}{\chi_{N+k}} p_{N+k}(w) = \frac{1}{\mathcal{D}_{N+k-1}} \begin{vmatrix} \oint_{\Sigma} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^{N+k} f(s) \frac{ds}{2\pi i s} \\ \oint_{\Sigma} s^{-1} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^{N+k-1} f(s) \frac{ds}{2\pi i s} \\ \vdots & & \vdots \\ \oint_{\Sigma} s^{-N-k+1} f(s) \frac{ds}{2\pi i s} & \cdots & \oint_{\Sigma} s^1 f(s) \frac{ds}{2\pi i s} \\ 1 & \cdots & w^{N+k} \end{vmatrix},$$

where $\mathcal{D}_{N+k-1} = \det(\oint_{\Sigma} s^{i-j} f(s) \frac{ds}{2\pi i s})_{i,j=0}^{N+k-1}$ and a similar expression exists for $Y_{21}(w)$, namely it equals the polynomial $-\chi_{N+k-1} w^{N+k-1} q_{N+k-1}(w^{-1})$. In particular, the uniqueness of the solution to the R -RHP, which then implies the uniqueness of the solution to Y -RHP guarantees that $\mathcal{D}_{N+k-1} \neq 0$. Now all of the entries appearing in this determinant as well as \mathcal{D}_{N+k-1} are analytic functions of γ so we conclude that Y_{11} (and similarly other entries of Y) are analytic functions of γ . Then, going back to R , we conclude that R is an analytic function of γ .

Now to obtain (4.22), we write for a fixed $\gamma \in [0, \gamma_0]$, L_γ for a square of side length ϵ centered at γ . Let us write also $R(w, \gamma)$ to highlight the dependence on γ . We note that by analyticity (Cauchy's integral formula), we have

$$\partial_\gamma R(w, \gamma) = \frac{1}{2\pi i} \oint_{L_\gamma} \frac{R(w, \mu)}{(\mu - \gamma)^2} \frac{d\mu}{2\pi i} = \frac{1}{2\pi i} \oint_{L_\gamma} \frac{R(w, \mu) - I}{(\mu - \gamma)^2} \frac{d\mu}{2\pi i}.$$

The first estimate in (4.22) then follows from the first estimate in (4.21). The second claim is similar and uses again the expression $\lim_{w \rightarrow \infty} (w(R(w, \gamma) - I)) = -\oint_{\Gamma_R} R_-(s, \gamma) \Delta_R(s, \gamma) \frac{ds}{2\pi i}$. For this, we also need an estimate for $\partial_\gamma \Delta_R(s, \gamma)$. This also can be estimated with a similar Cauchy integral formula argument due to the analyticity in γ , and the claim follows from our bounds on $\Delta_R(s, \gamma)$. This concludes the proof. \square

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